Generalizations of solid inflation

Jonghee Kang

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Abstract

Solid inflation is a unique inflationary model, in which inflatons have time-independent but spatially dependent vacuum expectation values. Since it does not conform to conventional inflationary models, it gives quite unique observational predictions, which in principle can be tested by observations. However, the original version of solid inflation hypothesizes an ideal type of solid: an isotropic solid. As a generalization, this thesis discusses a more realistic solid, which has a symmetry under a point group. As a result, its underlying structure can be maximally anisotropic even though it can still give isotropic predictions at the background and quadratic fluctuations in scalar modes. In another branch of generalizations, this thesis performs a thorough analysis of higher-derivative interactions in solid inflation, which the original version ignores.

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Acknowledgments

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1 Introduction

Since the discovery of the expansion of the universe by Hubble, Big Bang theory has been the standard model of cosmology. In particular, the Lambda cold dark matter (ΛCDM) model [1] has successfully passed many tests against the observational data. Despite its huge success, the ΛCDM model is incomplete in the sense that it lacks an explanation for the initial conditions. More importantly, it cannot solve the following three puzzles.

First, the Cosmological Microwave Background (CMB) observations tell us that the spatial curvature parameter in the Friedmann-Robertson-Walker (FRW) metric was very small at the time of last scattering. This spatial curvature parameter is increasing as the universe cools down, which means that in the early universe it should be even less than the value at last scattering, which is already very small. Strictly speaking, it is not a problem to have exactly zero spatial curvature. However, physicists usually believe that very small or zero curvature is not technically natural and demand a deeper explanation.

Secondly, the horizon at last scattering subtends only a few degrees, which poses another puzzle: why are the CMB and large-scale structures very close to homogeneous and isotropic? Two regions separated by more than the horizon at last scattering could not be causally connected in the past; therefore, there is no apparent explanation for the smoothing out of possible inhomogeneities.

Lastly, most grand unification theories unifying the strong nuclear interaction and electroweak interaction predict the existence of magnetic monopoles, which we have not observed so far. If they existed, it could have non-trivial cosmological implications, but cosmological models without magnetic monopoles match observations well.

It seems that these three puzzles are independent of each other, and at a first glance there does not seem to be one simple solution. Surprisingly it turns out that

[1] We commonly use the symbol Λ for the cosmological constant.

a single scenario can solve all of them successfully and straightforwardly. This is inflation (See [1–4] for the original papers and [5–7] for the reviews.). Generally speaking, inflation refers to an accelerated expanding era in the very early universe, which is around 10^{-33} and 10^{-32} seconds after Big Bang. The simplest scenario one may imagine for an accelerating expansion is de Sitter space. In de Sitter space, we have only a positive cosmological constant and nothing else. Then, the Hubble constant does not change as time goes on. It is proportional to $\sqrt{\Lambda}$ and the scale factor is

$$a(t) = e^{Ht}\,. \tag{1.1}$$

However, this is not what we actually need for inflation. Inflation should end and evolve into an ordinary ΛCDM phase, whereas an accelerating expansion is persistent in de Sitter space. Therefore, we need to come up with a dynamical model which makes inflation last only for a finite, but sufficiently long time. There are many models that have been proposed (see [8–28]), most of which are driven by single or multiple scalar fields, which are called inflaton fields. As a general review, let us consider the simplest model driven by a single scalar field, for which the action is

$$S = \int d^4x\,\sqrt{-g}\left[\frac{M_{\rm p}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\,. \tag{1.2}$$

with the $(-,+,+,+)$ signature. By varying the action with respect to the metric and ϕ, we obtain

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right)\,, && (1.3)\\ \rho &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\,, && (1.4)\\ p &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, && (1.5)\end{aligned}$$

and

$$-\nabla_\mu\nabla^\mu\phi + V'(\phi) = 0\,. \tag{1.6}$$

In general, a scalar field ϕ can take any form. However, let us consider a spatially independent but time-dependent background, $\phi(t,x) = \bar{\phi}(t)$. The reason for this choice is that our universe is homogeneous and isotropic at large scales. Hence, apart from small fluctuations around it, one may assume that the scalar field is homogeneous and isotropic. Then, on the background, spatial derivatives of the inflaton vanish and the above quantities and equation of motion become

$$\rho = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}), \tag{1.7}$$

$$p = \frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi}), \tag{1.8}$$

and

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\bar{\phi}) = 0. \tag{1.9}$$

The equation of state is defined as $\omega = p/\rho$, so the above background density and pressure give

$$\omega = \frac{\frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi})}{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})}, \tag{1.10}$$

which is generally time-dependent and bounded by $-1 \leq \omega \leq 1$ (for positive definite $V(\phi)$). From the Friedmann equations, if we want just a mildly accelerating expansion, having $\omega < -1/3$ is enough, or equivalently

$$\dot{\bar{\phi}}^2 < V(\bar{\phi}). \tag{1.11}$$

However, we would like to assert that during inflation the universe is close to de Sitter space. Therefore, we would like to impose a stronger limit on the equation of state, $\omega \simeq -1$, which corresponds to

$$\dot{\bar{\phi}}^2 \ll |V(\bar{\phi})|, \tag{1.12}$$

meaning the potential $V(\phi)$ should dominate over the kinetic term. Furthermore, we assume that the fractional change in $\dot{\bar{\phi}}$ during the Hubble time $1/H$ is very small [2] because we want inflation to be sustained for a sufficient period of time to solve the three puzzles above. That is,

$$\frac{|\ddot{\bar{\phi}}|}{H|\dot{\bar{\phi}}|} \ll 1. \tag{1.13}$$

[2] The Hubble time is approximately equal to the time for inflationary expansion.

Under (1.12) and (1.13), the equation of motion for a scalar field ϕ (1.9) reduces to

$$3H\dot{\bar{\phi}} + V'(\bar{\phi}) \simeq 0\,. \tag{1.14}$$

So far, we are working directly on the dynamics of a scalar field to construct a viable inflationary model. However, it may be more instructive to see what is going on from the perspective of geometry. Using the Friedman equations, the background density and pressure are related to the Hubble constant and its time derivative as follows:

$$H^2 = \frac{1}{3M_{\mathrm{p}}^2}\left(\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})\right)\,, \tag{1.15}$$

$$\dot{H} = -\frac{1}{2M_{\mathrm{p}}^2}\dot{\bar{\phi}}^2\,. \tag{1.16}$$

Using these H and $\dot{H}$, let us now define the two quantities

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \tag{1.17}$$

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H}\,. \tag{1.18}$$

These are, respectively, the measures of the fractional changes in H and ϵ during the Hubble time $1/H$. We would like both of their magnitudes to be very small, since the smallness of ϵ implies that the universe is expanding at an accelerating, even nearly exponential, rate, and the smallness of η implies that such expansion lasts for a sufficiently long time. This turns out to be the case if we impose (1.12) and (1.13), so that

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\dot{\bar{\phi}}^2}{\frac{1}{2}\dot{\bar{\phi}}^2 + V} \simeq \frac{3}{2}\frac{\dot{\bar{\phi}}^2}{V} \ll 1\,, \tag{1.19}$$

and

$$|\eta| = \left|\frac{\dot{\epsilon}}{\epsilon H}\right| = \left|\frac{\ddot{H}}{H\dot{H}} - \frac{2\dot{H}}{H^2}\right| = \left|\frac{\ddot{\bar{\phi}}}{2H\dot{\bar{\phi}}} + 2\epsilon\right| \ll 1\,. \tag{1.20}$$

We call ϵ and η *the slow-roll parameters* and their smallness *the slow-roll approximations*. The slow-roll approximations are the most important conditions to construct viable inflationary models, even though their smallness is not supposed to be persistent just as inflation itself is not. In other words, when (1.19) and (1.20) are violated, that is the moment when inflation ends. After inflation ends, the universe should enter the standard ΛCDM phase via reheating. Reheating refers to a general mechanism by which the inflaton field ϕ decays into ordinary radiation and matter we are familiar with. This process is not well understood yet, and we will not review it in this thesis (See [29] and references therein).

Now let us get back to the dynamics of a scalar field ϕ. As proved in Appendix A, the smallness of ϵ and η defined above is equivalent to the smallness of the following parameters:

$$\epsilon_V \equiv \frac{M_{\rm p}^2}{2}\left(\frac{V'(\bar{\phi})}{V(\bar{\phi})}\right)^2 , \tag{1.21}$$

$$\eta_V \equiv M_{\rm p}^2 \frac{V''(\bar{\phi})}{V(\bar{\phi})} . \tag{1.22}$$

The smallness of these two parameter suggests that the shape of the scalar potential $V(\phi)$ should be flat enough that the background of the inflaton, $\bar{\phi}$, slowly moves toward the minimum. See the below figure.

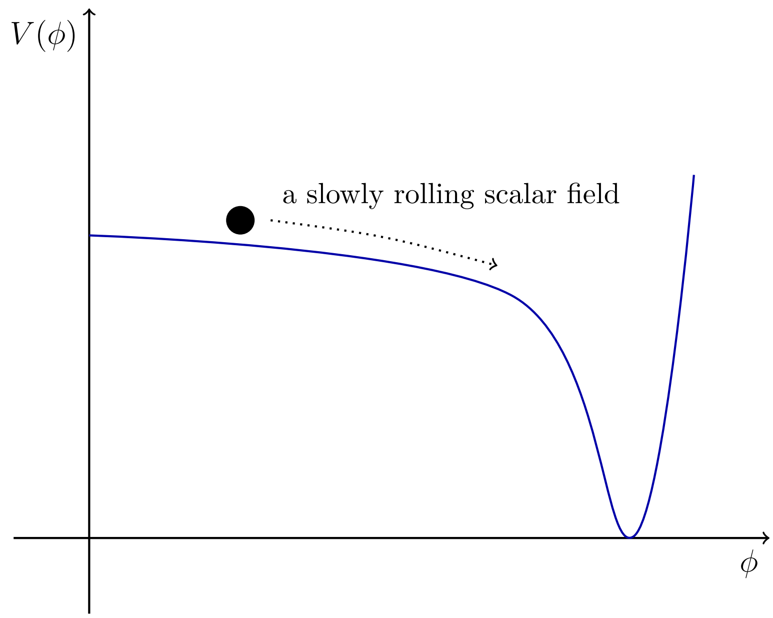


Figure 1: *The shape of a typical slow-roll potential, $V(\phi)$.*

The scalar fields rolls down to the global minimum from the origin. Since the potential is almost flat near the starting point, it takes a long time to get to the minimum, which makes inflation sustained for as long a time as we wish. This is why we call the two parameters ϵ and η the slow-roll parameters.

We now have a complete setup for inflation to solve the three puzzles of standard Big Bang theory. In fact, this is not the end of the story. The power of inflation exceeds just solving those three puzzles. So far, we have discussed only the background dynamics of the inflaton to see the expansion of the universe. As mentioned above, the CMB and large-scale structures observations tell us that the universe has small inhomogeneities in addition to it being almost homogeneous and isotropic at large scales. If the universe were perfectly homogeneous and isotropic, then neither any astronomical objects nor even we could exist. However, introducing small inhomogeneities around a homogeneous background is not easy without any plausible explanation. Many conundrums in modern physics are exactly like this (see [30–36] for the cosmological constant problem and see [37–43] for the hierarchy problem of the Higgs mass). Again, this tuning is not a problem in itself. It is perfectly fine to have such a delicate initial condition and there is no logical flaw in having it. However, it turns out that inflation can also solve this fine-tuned initial condition problem beautifully based on one simple quantum mechanical principle: quantum fluctuation. Around the background $\phi(x) = \bar{\phi(t)}$, a scalar field ϕ can have a quantum mechanically generated fluctuation, $\delta\phi(x)$. Due to this fluctuation, not all patches in the universe are inflated by the same amount; some are inflated more and others are inflated less. During inflation, these inhomogeneities are stretched across the horizon scale and thus smoothed out. These stretched modes are the seeds of initial small inhomogeneities and have grown since the end of inflation due to gravitational instability. The analysis of these perturbations can be done analytically, and more importantly, this analysis is what connects to the observational data. Quantum fluctuations are inherently stochastic, so we need to compute their correlation functions to make contact with observations. Since the background is isotropic, we can decompose any mode into scalar, vector and tensor components. At quadratic order, they are not coupled to each other, so we may focus on the correlations between one single submode [3].

Before we do actual computations, we need to spend a bit more time on the setup. First of all, the theory of general relativity is a gauge theory, so we have gauge freedom. Therefore, it is important to make sure that what we compute is gauge invariant. Unfortunately, not every quantity we have considered so far is gauge invariant. For instance, $\delta\phi(x)$ is not gauge invariant, and neither is the correlation function of $\delta\phi(x)$. Therefore, the correlation function of $\delta\phi(x)$ is not an appropriate object to match with observations, and we have to find the gauge invariant combinations that contain information about $\delta\phi(x)$. Out of various scalar modes in the metric

[3] We will not review the correlation functions of a vector mode, since a vector mode generally decays.

and stress-energy tensor, there are two important gauge-invariant combinations,

$$\mathcal{R} = \frac{A}{2} + H\delta u\,, \tag{1.23}$$

$$\zeta = \frac{A}{2} - H\frac{\delta\rho}{\dot{\bar{\rho}}}\,, \tag{1.24}$$

where the perturbed metric and stress-energy tensor before gauge fixing [4] are

$$\delta g_{00} = -E\,, \tag{1.25}$$

$$\delta g_{i0} = a(\partial_i F + G_i)\,, \tag{1.26}$$

$$\delta g_{ij} = a^2(A\delta_{ij} + \partial_i\partial_j B + \partial_i C_j + \partial_j C_i + D_{ij})\,, \tag{1.27}$$

and

$$\delta T_{00} = -\bar{\rho}\delta g_{00} + \delta\rho\,, \tag{1.28}$$

$$\delta T_{i0} = \bar{p}\delta g_{i0} - (\bar{\rho} + \bar{p})\left(\partial_i\delta u + \delta u_i^V\right)\,, \tag{1.29}$$

$$\delta T_{ij} = \bar{p}\delta g_{ij} + a^2\left(\delta p\delta_{ij} + \partial_i\partial_j\rho^S + \partial_i\rho_j^V + \partial_j\rho_i^V + \rho_{ij}^T\right)\,, \tag{1.30}$$

where G_i, C_i, δu_i^V and ρ_i^V are transverse vector modes and D_{ij} and ρ_{ij}^T are transverse and traceless tensor modes. ζ and $\mathcal{R}$ are usually called the curvature perturbations because, apart from an overall coefficient and ∇^2, they are equal to the perturbation of the three-dimensional Ricci scalar associated with a perturbed g_{ij},

$$R^{(3)} = -\frac{4}{a^2}\nabla^2\left(\frac{A}{2}\right) + \cdots\,, \tag{1.31}$$

in the corresponding famous gauge choices [5]. Therefore, instead of computing a two-point function of $\delta\phi$, we will compute a two-point function of ζ (or $\mathcal{R}$) [6].

After going through computations which we will not review here, the two-point

[4] We follow the convention in [44].

[5] The comoving gauge where $\delta u = 0$ for $\mathcal{R}$ and the constant density gauge where $\delta\rho = 0$ for ζ.

[6] When there are only adiabatic modes present, ζ and $\mathcal{R}$ are equal at superhorizon scales.

functions of scalar and tensor modes are as follows [7]:

$$\begin{aligned}
\langle \zeta_k \zeta_{k'} \rangle &= (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{2\pi^2}{k^3} P_s(k)\,, && (1.32) \\
P_s(k) &= \frac{H_0^2}{8\pi^2 \epsilon_0 M_{\mathrm{p}}^2}\,, && (1.33) \\
\langle \gamma_k^s \gamma_{k'}^{s'} \rangle &= (2\pi)^3 \delta(\vec{k} + \vec{k}') \delta_{s,s'} \frac{2\pi^2}{k^3} P_\gamma(k)\,, && (1.34) \\
P_\gamma(k) &= \frac{2H_0^2}{\pi^2 M_{\mathrm{p}}^2}\,. && (1.35)
\end{aligned}$$

The power spectra of the scalar and tensor modes are not perfectly scale invariant because some k dependence is hidden in the Hubble constant and the slow-roll parameters. The measures of deviation from scale invariance are defined as

$$n_s - 1 \equiv \frac{d \ln P_s(k)}{d \ln k} = -2\epsilon_0 - \eta_0\,, \qquad (1.36)$$

$$n_\gamma - 1 \equiv \frac{d \ln P_\gamma(k)}{d \ln k} = -2\epsilon_0\,. \qquad (1.37)$$

n_s and n_γ are called the scalar and tensor tilts, respectively, and they are very important observable predictions of inflation. The fact that they are suppressed by the slow-roll parameters suggests that primordial power spectra of inflation are nearly scale invariant, which is well supported by observational data. Another observable we have is the relative order of magnitude of the scalar and tensor modes, defined in the following way:

$$r \equiv \frac{P_\gamma}{P_s} = 16\epsilon_0\,, \qquad (1.38)$$

which is called the tensor-to-scalar ratio.

Currently, the best measured values for n_s and r are believed to follow from the

[7] The subscript 0 on the Hubble constant and the slow-roll parameters denote they are evaluated at the horizon crossing.

Planck 2015 result [8] [45],

$$n_s = \begin{cases} 0.9655 \pm 0.0062 \quad (68\% \text{ CL, Planck TT} + \text{lowP}), \\ 0.9645 \pm 0.0049 \quad (68\% \text{ CL, Planck TT, TE, EE} + \text{lowP}), \\ 0.9677 \pm 0.0060 \quad (68\% \text{ CL, Planck TT} + \text{lowP} + \text{lensing}), \end{cases} \tag{1.39}$$

$$r < \begin{cases} 0.103 \quad (95\% \text{ CL, Planck TT} + \text{lowP}), \\ 0.099 \quad (95\% \text{ CL, Planck TT, TE, EE} + \text{lowP}), \\ 0.114 \quad (95\% \text{ CL, Planck TT} + \text{lowP} + \text{lensing}). \end{cases} \tag{1.40}$$

The observables introduced thus far are based on dynamics up to quadratic order in fluctuations. In fact, more interesting features which can distinguish between different inflationary models emerge when we expand the action up to cubic order and compute higher-order correlation functions. The next-order correlation function is the three-point function, also known as the trispectrum in Fourier space. As we mentioned above, in contrast to correlation functions in ordinary quantum field theory, what we compute are in-in correlation functions [46]. Therefore, we have to compute an expectation value in the presence of a cubic action, which is treated as an interaction [47],

$$\langle \zeta^3(\tau) \rangle = \langle \Omega(-\infty) | U_{int}^{-1}(\tau, -\infty) \zeta^3(\tau) U_{int}(\tau, -\infty) | \Omega(-\infty) \rangle \,, \tag{1.41}$$

where

$$U_{int}(\tau, -\infty) = T\, e^{-i \int_{-\infty}^{\tau} d\tau'\, H_{int}(\tau')} \,. \tag{1.42}$$

The detailed form of a three-point function is highly model-dependent. However, there is a model-independent way to measure the overall scale of a three-point function relative to a two-point function via the Newtonian potential Φ [48]:

$$\Phi = \tfrac{3}{5} \zeta \,. \tag{1.43}$$

[8]In the single-field inflationary models, we always have $n_\gamma - 1 \simeq -\frac{r}{8}$ at leading order in the slow-roll limit, so for n_γ, we can refer to the value for r.

Parametrizing the two- and three-point functions of Φ as:

$$\left\langle \Phi(\vec{k_1})\Phi(\vec{k_2})\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\frac{\Delta_\Phi}{k_1^3}\,, \tag{1.44}$$
$$\left\langle \Phi(\vec{k_1})\Phi(\vec{k_2})\Phi(\vec{k_3})\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_2)f(k_1,k_2,k_3)\,, \tag{1.45}$$

one defines $f_{\rm NL}$ using equilateral configurations with $|\vec{k}_1| = |\vec{k}_2| = |\vec{k}_3| \equiv k$;

$$f(k,k,k) = f_{\rm NL}\frac{6\Delta_\Phi^2}{k^6}\,. \tag{1.46}$$

1.1 The effective field theory of inflation

1.1.1 Versatility of inflation

One huge advantage of inflation, which may at the same time be a huge disadvantage, is its versatility. We can come up with many (perhaps too many) inflationary models by considering different potentials $V(\phi)$, introducing more than one scalar field, introducing non-minimal coupling with gravity, etc, as long as the two slow-roll conditions are met. Of course, there are a few physical constraints and consistency relations that an inflationary model has to satisfy, so building it is not arbitrary. However, an inflationary model can be tuned to fit a vast region of the parameter space. See the figure below [9].

The main disadvantage of this feature is that it is hard to narrow down the possibilities to a single inflationary model, which actually describes the true inflationary expansion. At the same time, it is hard to come up with a unified framework to make comparison between the observations and the set of concrete theoretical predictions.

[9] The figure is taken from [45].

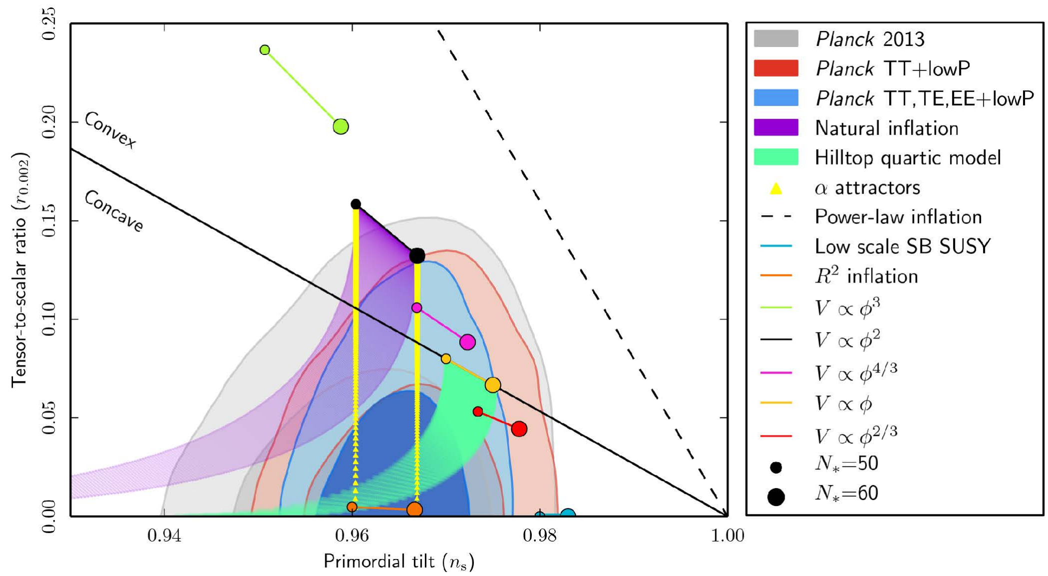


Figure 2: *Marginalized joint 68 % and 95 % CL regions for* n_s *and* r *at* $k = 0.002Mpc^{-1}$ *from Planck compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions have been obtained by assuming* $dn_s/d\,lnk = 0$.

1.1.2 Effective field theory approach

To overcome this shortcoming of inflation, the authors of [49] suggested a novel approach: despite the many forms a self-interacting potential $V(\phi)$ can take, the existence of more than one scalar field, and the existence of non-minimal coupling to gravity, almost all inflationary models and, more generally, all cosmological models share one common feature; they have a homogeneous and isotropic background. In the language of single-field inflation, it can be stated as the condition that we have a time-dependent but spatially independent scalar field and metric on the background:

$$\phi(t,x) = \bar{\phi}(t)\,, \tag{1.47}$$

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2\,. \tag{1.48}$$

We are familiar with a similar situation in quantum field theory, when a theory has some set of symmetries, but the vacuum does not. Of course, this is a system whose symmetries are spontaneously broken. In the case of inflation, since gravity couples to the inflaton field (or fields), the theory must have full diffeomorphism invariance. However, due to the time-dependent backgrounds, time diffeomorphisms are spontaneously broken. Because of the well-known Goldstone theorem, there should exist a Goldstone mode associated with each broken symmetry. That is, we have one Goldstone mode associated with the broken time diffeomorphism symmetry. This Goldstone mode transforms non-linearly with respect to the broken time diffeomorphisms and is actually related to the fluctuation of the scalar field, $\delta\phi(x)$, in the original language.

The rule of thumb of the effective field theory is to write down all possible terms in the action which are consistent with the symmetries of our action. Handling the broken symmetries requires some caution though, since the broken symmetries are non-linearly realized, as we just mentioned. To get around this issue, we can utilize the fact that diffeomorphisms are the gauge symmetries of general relativity. Since the broken symmetries are gauge symmetries, we can choose the "*unitary gauge*" in which there are no Goldstone modes, i.e. $\phi(t,x) = \bar{\phi}(t)$. In other words, the Goldstone mode is eaten by the gauge fields, which consequently become massive. Of course, this is the well-known Higgs mechanism [50, 51]. By doing so, we deal with time diffeomorphisms, and the only remaining condition we need to satisfy is that all terms in our action be invariant under the unbroken symmetries, spatial diffeomorphisms. Notice that the original Einstein-Hilbert action is invariant under the full diffeomorphism symmetry. In addition, we can introduce terms that are invariant under just spatial diffeomorphisms, but not under time diffeomorphisms. In "unitary gauge," there is no Goldstone mode and we have only the metric and curvature tensors as our building blocks. Out of these, we can build the following action up to quadratic order in fluctuations [10]:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm p}^2}{2} R - c(t) g^{00} - \Lambda(t) + \frac{1}{2!} M_2(t)^2 (\delta g^{00})^2 \right. \\ \left. - \frac{1}{2} \bar{M}_1(t)^3 \delta g^{00} \delta K^\mu{}_\mu - \frac{1}{2} \bar{M}_2(t)^2 \delta K^\mu{}_\mu{}^2 - \frac{1}{2} \bar{M}_3(t)^2 \delta K^\mu{}_\nu \delta K^\nu{}_\mu + \cdots \right] , \tag{1.49}$$

where $\delta K^\mu{}_\nu$ is the perturbation of the extrinsic curvature of a constant time surface. Except for the first three terms, which do not vanish even on the background, we parametrize all other terms by δg^{00} and $\delta K^\mu{}_\nu$. By doing so, it is evident which terms we should keep to perform a perturbative analysis up to a certain order. Moreover, the background values of g^{00} and $K_{\mu\nu}$ [11] are also invariant under spatial diffeormophisms, so we can have any polynomials in those quantities as long as indices are contracted properly.

As we just mentioned, only the first three terms in (1.49) survive on the background and govern the background dynamics. Therefore, the coefficients $c(t)$ and $\Lambda(t)$, which are generic functions of time *a priori*, are actually related to the Hubble constant via the Friedmann equations. Those relations are

$$\begin{aligned} c(t) &= -M_{\rm p}^2 \dot{H} \, , & (1.50) \\ \Lambda(t) &= M_{\rm p}^2 (3H^2 + \dot{H}) \, . & (1.51) \end{aligned}$$

Therefore, we can rewrite (1.49) as follows:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm p}^2}{2} R + M_{\rm p}^2 \dot{H} g^{00} - M_{\rm p}^2 (3H^2 + \dot{H}) + \frac{1}{2!} M_2(t)^2 (\delta g^{00})^2 \right. \\ \left. - \frac{1}{2} \bar{M}_1(t)^3 \delta g^{00} \delta K^\mu{}_\mu - \frac{1}{2} \bar{M}_2(t)^2 \delta K^\mu{}_\mu{}^2 - \frac{1}{2} \bar{M}_3(t)^2 \delta K^\mu{}_\nu \delta K^\nu{}_\mu + \cdots \right] , \tag{1.52}$$

In (1.52), there are three degrees of freedom: two of them are the ordinary graviton helicities and one is the Goldstone mode, which is a scalar and is eaten by the gauge fields. Actually, the dynamics of that Goldstone mode generates observable signals, so we would like to restore it in (1.52) by the Stückelberg mechanism. The basic idea

[10] We can add higher-order terms such as $\frac{1}{3!} M_3(t)^4 (\delta g^{00})^3$ if we are interested.

[11] $g^{00} = -1$ and $K_{\mu\nu} = a^2 H h_{\mu\nu}$ on the FRW background, where $h_{\mu\nu}$ is the induced spatial metric.

is the following: Apply a time diffeomorphism, $t \to \tilde{t} \equiv t + \xi^0(x)$, which is our broken symmetry. Then, wherever $\xi^0(x)$ appears, replace $\xi^0(x)$ with a field $\pi(x)$, which transforms non-linearly as $\pi(x) \to \pi(x) - \xi^0(x)$ under the time diffeomorphism. For instance, we do the following replacements:

$$t \to t + \pi(x)\,, \tag{1.53}$$

and

$$\begin{aligned} g^{00} \to & \frac{\partial(t+\pi(x))}{\partial x^\mu}\frac{\partial(t+\pi(x))}{\partial x^\nu} g^{\mu\nu} = (\delta^0_\mu + \partial_\mu \pi(x))(\delta^0_\nu + \partial_\nu \pi(x)) g^{\mu\nu} \\ & = (1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i \pi g^{0i} + \partial_i \pi \partial_j \pi g^{ij}\,, \end{aligned} \tag{1.54}$$

because under the time diffeomorphism, $t \to t + \xi^0(x)$ and $g^{00} \to \frac{\partial(t+\xi^0(x))}{\partial x^\mu}\frac{\partial(t+\xi^0(x))}{\partial x^\nu} g^{\mu\nu}$. After making these replacements, our action is now also invariant under broken time diffeomorphisms even though they are non-linearly realized by the Goldstone mode $\pi(x)$. Ignoring higher-order derivative terms, the new action is

$$\begin{aligned} S = \int d^4x \sqrt{-g} \Bigg[& \frac{M_\mathrm{p}^2}{2} R - M_\mathrm{p}^2 (3H^2(t+\pi(x)) + \dot{H}(t+\pi(x))) \\ & + M_\mathrm{p}^2 \dot{H}(t+\pi(x)) \left((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i \pi g^{0i} + \partial_i \pi \partial_j \pi g^{ij}\right) \\ & + \frac{1}{2!} M_2(t)^2 \left(1 + (1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i \pi g^{0i} + \partial_i \pi \partial_j \pi g^{ij}\right)^2 + \cdots \Bigg]\,. \end{aligned} \tag{1.55}$$

The only difference between (1.52) and (1.55) is that in (1.52) the scalar Goldstone mode is eaten by the metric and everything is written in terms of the metric, while in (1.55) the scalar goldstone mode is around explicitly and the metric remains massless. At a first glance, (1.55) just seems to be creating more complications, however as shown in [49], we can simplify it under physically meaningful limits.

1.2 Solid inflation

The effective field theory of inflation encompasses most inflationary scenarios, but there can be exceptions. The authors of [52] suggested a new inflationary model driven by scalar fields that have time-independent but spatially dependent backgrounds. The dynamics of the scalar fields they introduced are equivalent to that of a cosmological solid, so they called the model "solid inflation." In the formalism of solid inflation, the authors still relied on the framework of effective field theory. However, the crucial difference from the effective field theory of inflation is that solid inflation does not have the same underlying symmetries or symmetry breaking pattern. In solid inflation, the scalar fields now have time-independent but spatially dependent vacuum expectation values, so spatial diffeomorphisms are now spontaneously broken symmetries.

From an effective field theory standpoint, the mechanical deformations of an homogeneous solid can be described in terms of three scalar fields $\phi^I(x)$ $(I = 1, 2, 3)$ [53], whose expectation values in the ground state of the solid are

$$\langle \phi^I \rangle = x^I \tag{1.56}$$

and whose Lagrangian is invariant under the shift symmetries

$$\phi^I \to \phi^I + a^I \,, \qquad a^I = \text{const} \tag{1.57}$$

(see also [54,55] for alternative approaches.) The ϕ^I's can be regarded as the comoving coordinates of the solid's volume elements. By Poincaré- and shift-invariance, to lowest order in derivatives the Lagrangian must take the form

$$\mathcal{L} = F\big(B^{IJ}\big) \,, \qquad B^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J \,, \tag{1.58}$$

where F is an *a-priori* generic function, determined by the solid's equation of state. For a solid with symmetry group $G \subset SO(3)$, one also demands that the Lagrangian be invariant under the internal rotations

$$\phi^I \to O^I{}_J \phi^J \,, \qquad O^I{}_J \in G \,, \tag{1.59}$$

which restrict the form of F.

For instance, in the case in which G is the full $SO(3)$—which is the case extensively studied in [52]—F can only depend on three invariants, e.g.

$$[B]\,, \qquad [B^2]\,, \qquad [B^3]\,, \tag{1.60}$$

where the square brackets denote the trace of the matrix within. Any other rotationally invariant function of B^{IJ} can be expressed in terms of these, e.g.

$$\det B = \tfrac{1}{6}\big([B]^3 - 3[B][B^2] + 2[B^3]\big)\,. \tag{1.61}$$

Upon minimally coupling the solid to gravity, the form of F is restricted further by demanding that the solid be able to drive near exponential inflation. In order for that to happen, one needs a solid that can be stretched by a large exponential factor without changing too much its physical properties, such as its energy density. Such a behavior is of course unlike that of any standard solid we know of, but it can be achieved by imposing an approximate *internal* scale invariance [52]:

$$\phi^I \to \lambda\,\phi^I\,. \tag{1.62}$$

Focusing again on the $SO(3)$ invariant case, to implement this symmetry it is useful to organize the three invariants (1.60) as

$$X = [B]\,, \qquad Y = \frac{[B^2]}{[B]^2}\,, \qquad Z = \frac{[B^3]}{[B]^3}\,. \tag{1.63}$$

X depends on the overall normalization of B, but Y and Z do not, and as a result Y and Z are invariant under the transformation (1.62). The requirement of approximate scale invariance thus translates into a weak dependence of $F(X,Y,Z)$ on X. In particular, by evaluating the solid's stress-energy tensor on an FRW background, one finds [52]

$$\begin{aligned} T_{\mu\nu} \;=\; & -2\partial_\mu\phi^I\partial_\nu\phi^J\left[\left(F_X - \frac{2F_YY}{X} - \frac{3F_ZZ}{X}\right)\delta^{IJ} + \frac{2F_YB^{IJ}}{X^2} + \frac{3F_ZB^{IK}B^{KJ}}{X^3}\right] \\ & + g_{\mu\nu}F(X,Y,Z) \end{aligned} \tag{1.64}$$

and

$$\rho = -F, \qquad p = F - \tfrac{2}{a^2} F_X \qquad\qquad (X = 3/a^2)\,, \tag{1.65}$$

which yields the slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{a^2}\frac{F_X}{F} = \frac{\partial \log F}{\partial \log X}\,. \tag{1.66}$$

Of course "slow-roll" here is a bad characterization, because nothing is rolling, slowly or otherwise: the background configurations for our ϕ^I's only depend on the *spatial* coordinates. But the slow-roll parameters like ϵ and the higher-order ones can also be defined geometrically, without any reference to rolling fields, purely in terms of the time-dependence of the Hubble scale H. We will adopt these geometric definitions—as we did above for ϵ—and still use the standard slow-roll nomenclature.

In the presence of perturbations, the scalars and the metric are

$$\phi^I = x^I + \pi^I(x)\,, \qquad g_{\mu\nu} = g^{\rm FRW}_{\mu\nu}(t) + \delta g_{\mu\nu}(x)\,, \tag{1.67}$$

where $g^{\rm FRW}_{\mu\nu} = {\rm diag}(-1, a^2, a^2, a^2)$ is the standard FRW metric. At distances much shorter than the Hubble radius, one can neglect the metric perturbations and identify $\vec{\pi}(x)$ with the phonon field. Expanding the solid Lagrangian to quadratic order, one finds two parameters c_L and c_T characterizing the longitudinal and transverse phonon propagation speed. These are determined by certain derivatives of F, evaluated on the background configuration:

$$c_L^2 = 1 + \frac{2}{3}\frac{F_{XX}X^2}{F_X X} + \frac{8}{9}\frac{F_Y + F_Z}{F_X X}\,, \tag{1.68}$$

$$c_T^2 = 1 + \frac{2}{3}\frac{F_Y + F_Z}{F_X X}\,. \tag{1.69}$$

In particular, one finds the universal exact relation [52]

$$c_T^2 = \tfrac{3}{4}\big(1 + c_L^2 - \tfrac{2}{3}\epsilon + \tfrac{1}{3}\eta\big)\,, \tag{1.70}$$

where $\eta \equiv \dot{\epsilon}/\epsilon H$ is the second slow-roll parameter.

To study cosmological perturbations and compute their correlation functions, it is convenient to decompose the metric in an Arnowitt-Deser-Misner (ADM) fashion,

$$ds^2 = -N^2 dt^2 + h_{ij}\big(dx^i + N^i dt\big)\big(dx^j + N^j dt\big)\,, \tag{1.71}$$

and choose spatially flat slice gauge (SFSG),

$$\phi^I = x^I + \pi^I\,, \qquad h_{ij} = a(t)^2 \exp(\gamma_{ij})\,, \qquad \partial_i \gamma_{ij} = \gamma_{ii} = 0\,. \tag{1.72}$$

Moreover, we can decompose N, N^i and π^i as follows:

$$N = 1 + \delta N\,, \qquad N^i = \frac{\partial_i}{\sqrt{-\nabla^2}} N_L + N_T^i\,, \qquad \pi^i = \frac{\partial_i}{\sqrt{-\nabla^2}} \pi_L + \pi_T^i\,. \tag{1.73}$$

The full action including the Einstein-Hilbert in SFSG gauge is now written as

$$S = \int d^4x\, N\sqrt{h}\left[\frac{M_{\rm p}^2}{2}\left(R^{(3)} + N^{-2}(E_{ij}E^{ij} - E^2)\right) + F(X,Y,Z)\right]\,, \tag{1.74}$$

where $E_{ij} = \frac{1}{2}\Big(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\Big)$, and $R^{(3)}$ and ∇_i are the three-dimensional Ricci scalar and covariant derivative constructed using h_{ij}, respectively. In (1.74), N and N^i are not dynamical so the equations of motion for them are just constraint equations:

$$\begin{aligned}
\frac{M_{\rm p}^2}{2}\left[R^{(3)} - N^{-2}(E_{ij}E^{ij} - E^2)\right] + F + N\frac{\partial F}{\partial N} &= 0\,, &(1.75)\\
M_{\rm p}^2 \nabla_i\left[N^{-1}(E^i_j - \delta^i_j E)\right] + N\frac{\partial F}{\partial N^j} &= 0\,. &(1.76)
\end{aligned}$$

The solutions of these two constraint equations in Fourier space are

$$\begin{aligned}
\delta N(t,k) &= -\frac{a^2\dot{H}}{kH}\frac{\dot{\pi}_L - \pi_L \dot{H}/H}{1 - 3a^2\dot{H}/k^2}\,, &(1.77)\\
N_L(t,k) &= \frac{-3a^2\dot{\pi}_L\dot{H}/k^2 + \pi_L\dot{H}/H}{1 - 3a^2\dot{H}/k^2}\,. &(1.78)
\end{aligned}$$

Plugging these two solutions back into (1.74) and expanding to quadratic order gives the following scalar and tensor actions:

$$\begin{aligned}
S_\gamma^{(2)} &= \frac{M_{\rm p}^2}{4}\int dt \int d^3x\, a^3\left[\frac{1}{2}\dot{\gamma}_{ij}^2 - \frac{1}{2a^2}(\partial_k\gamma_{ij})^2 + 2\dot{H}c_T^2\gamma_{ij}^2\right]\,, &(1.79)\\
S_s^{(2)} &= M_{\rm p}^2 \int dt \int d^3k\, a^3\left[\frac{k^2/3}{1 - k^3/3a^2\dot{H}}|\dot{\pi}_L - (\dot{H}/H)\pi_L|^2 + \dot{H}c_L^2 k^2|\pi_L|^2\right]\,. &(1.80)
\end{aligned}$$

As mentioned previously, we do not compute correlation functions of π_L, but those of the gauge invariant curvature perturbations. For the reasons spelled out in [52], we choose to use the curvature perturbation ζ, which in the above gauge is related to the phonon field $\vec{\pi}$ by

$$\zeta = \tfrac{1}{3}\vec{\nabla}\cdot\vec{\pi}\,. \tag{1.81}$$

Then, using standard cosmological perturbation theory, one can compute the correlation functions for scalar and tensor modes. At the two-point function level, the relevant observables are the scalar tilt, the tensor tilt, and the tensor-to-scalar ratio:

$$n_s - 1 \simeq 2\,\epsilon c_L^2 - 5s - \eta\,, \tag{1.82}$$

$$n_T - 1 \simeq 2c_L^2\epsilon\,, \tag{1.83}$$

$$r \simeq 16\,\epsilon c_L^5\,, \tag{1.84}$$

where s monitors the time-dependence of c_L, $s \equiv \dot{c}_L/c_L H$. Particularly unusual predictions are the positivity of the tensor tilt—which would usually require a violation of the null energy condition—and the strong suppression of the tensor-to-scalar ratio in the slow sound speed limit, a factor of c_L^4 stronger than for standard single-field cases.

Expanding further the solid Lagrangian to cubic order, one finds that at leading order in the slow-roll limit the phonon self-interactions take the form

$$\mathcal{L}^{(3)} = M_{\rm p}^2 a(t)^3 H^2 \frac{F_Y}{F}\Big\{\tfrac{7}{81}(\partial_i\pi^i)^3 - \tfrac{1}{9}\partial_i\pi^i\partial_j\pi^k\partial_k\pi^j - \tfrac{4}{9}\partial_i\pi^i\partial_j\pi^k\partial_j\pi^k + \tfrac{2}{3}\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i\Big\}. \tag{1.85}$$

The gravitational corrections on this are suppressed both in the de-mixing regime, $k \gg aH\epsilon^{1/2}$, and in the strong mixing one, $k \ll aH\epsilon^{1/2}$, and one can argue that the cubic Lagrangian above is all one needs to compute the three-point function of curvature perturbations [52].

To leading order in slow roll, the result is

$$\begin{aligned}\left\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\right\rangle &\simeq (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{3}{32}\frac{F_Y}{F}\frac{H^4}{M_{\rm p}^4}\frac{1}{\epsilon^3 c_L^{12}} \\ &\quad\times\frac{Q(\vec{k}_1,\vec{k}_2,\vec{k}_3)U(k_1,k_2,k_3)}{k_1^3k_2^3k_3^3} \end{aligned}\tag{1.86}$$

where

$$\begin{aligned}Q(\vec{k}_1,\vec{k}_2,\vec{k}_3) &\equiv \frac{7}{81}k_1k_2k_3-\frac{5}{27}\left(k_1\frac{(\vec{k}_2\cdot\vec{k}_3)^2}{k_2k_3}+k_2\frac{(\vec{k}_3\cdot\vec{k}_1)^2}{k_3k_1}+k_3\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{k_1k_2}\right) \\ &\quad+\frac{2}{3}\frac{(\vec{k}_1\cdot\vec{k}_2)(\vec{k}_2\cdot\vec{k}_3)(\vec{k}_3\cdot\vec{k}_1)}{k_1k_2k_3}\end{aligned}\tag{1.87}$$

and

$$\begin{aligned}U(k_1,k_2,k_3) &= \frac{2}{k_1k_2k_3\left(k_1+k_2+k_3\right)^3}\Big\{3\big(k_1^6+k_2^6+k_3^6\big)+20k_1^2k_2^2k_3^2 \\ &\quad+18\big(k_1^4k_2k_3+k_1k_2^4k_3+k_1k_2k_3^4\big)+12\big(k_1^3k_2^3+k_2^3k_3^3+k_3^3k_1^3\big) \\ &\quad 9\big(k_1^5k_2+5\,\text{perms}\big)+12\big(k_1^4k_2^2+5\,\text{perms}\big) \\ &\quad+18\big(k_1^3k_2^3k_3+5\,\text{perms}\big)\Big\}\,.\end{aligned}\tag{1.88}$$

Assuming $F_Y \sim F$, this has a potentially huge $f_{\rm NL}$,

$$f_{\rm NL} = -\frac{19415}{13122}\frac{F_Y}{F}\frac{1}{\epsilon c_L^2} \sim \frac{1}{\epsilon c_L^2}\,,\tag{1.89}$$

but its most peculiar feature is probably its 'shape' [56]—in particular, its purely quadrupolar angular dependence in the squeezed limit $k_3 \ll k_{1,2}$:

$$\langle\zeta\zeta\zeta\rangle \propto \frac{(1-3\cos^2\theta)}{k_1^3k_3^3}\,,\tag{1.90}$$

where θ is the angle between $\vec{k}_1$ and $\vec{k}_3$.

1.3 Outline

In Sect. 2 and 3, we will review the anisotropic generalization of the original solid inflation [52]. Specifically, in Sect. 2, we will identify an appropriate anisotropic structure and focus on its implications for scalar modes at cubic level. For scalar modes, cubic order is the lowest order at which unobserved anisotropic signals can exist. In Sect. 3, we will study the anisotropic generalizations of tensor modes, now at quadratic level which is the lowest order at which anisotropic signals can exist. As we will see in Sect. 3, the anisotropic tensor kinetic term can be present only when higher-order derivative interactions are introduced. Therefore, we will perform a systematic analysis of such higher-order derivative interactions and what their existence implies in Sect. 4.

This thesis is largely based on [57–59]. In particular, Sect. 1 and Sect. 2 are based on [57], Sect. 3 is based on [58] and Sect. 4 is based on [59].

2 Introduction of maximal anisotropy to a solid

The universe appears isotropic on large scales, and it is thus natural to assume that whatever it was that fueled primordial inflation, it was an isotropic system. It is interesting, however, to analyze critically this assumption. Observations tell us that the cosmological background and the spectrum of scalar perturbations are isotropic, but they do not tell us anything about higher-point correlation functions or about tensor modes, for the simple reason that we have not detected them yet.

This raises the following question: can one have a physical system driving inflation whose dynamics are intrinsically anisotropic—perhaps maximally so—but that nevertheless yields an isotropic background and an isotropic scalar spectrum of perturbations? We are not interested in systems for which one can tune coefficients in the Lagrangian in order to achieve the desired degree of isotropy, but rather in systems whose symmetries are so powerful as to enforce such an isotropy, leaving open the possibility of anisotropic signals for other observables.

To make the discussion more concrete, let us consider the cubic group. This is the discrete subgroup of rotations that maps a cubic lattice into itself. Calling $\hat{x}$, $\hat{y}$, $\hat{z}$ the lattice's preferred directions, the cubic group is simply the set of permutations of $\hat{x}$, $\hat{y}$, $\hat{z}$ as well as single-axis inversions $\hat{x} \to -\hat{x}$, etc. Barring fine-tunings, the dynamics of a homogenous system with this symmetry group—such as a cubic crystal in the continuum limit—in general will not be isotropic. However, certain observables are forced to be. In particular, because of cubic symmetry, any two-index tensor associated with the lattice must take the form

$$T^{ij} \propto \hat{x}^i \hat{x}^j + \hat{y}^i \hat{y}^j + \hat{z}^i \hat{z}^j = \delta^{ij} \, . \tag{2.1}$$

On the other hand, with more indices there are structures that are invariant under the cubic group only, and can in principle lead to observable anisotropies. For instance, at the four-index level, the tensor structure

$$\hat{x}^i \hat{x}^j \hat{x}^k \hat{x}^l + \hat{y}^i \hat{y}^j \hat{y}^k \hat{y}^l + \hat{z}^i \hat{z}^j \hat{z}^k \hat{z}^l \tag{2.2}$$

is invariant under the cubic group, but cannot be rewritten in terms of Kronecker deltas only.

Solids are natural candidates for considering discrete subgroup of rotations, and for this reason we will elaborate on the above ideas in the context of solid inflation [52] [12]. In solid inflation, inflation is driven by a solid's stress-energy tensor $T^{\mu\nu}$. For the background evolution to be isotropic, one needs an isotropic T^{ij} on the ground state of the solid. However, T^{ij} in general is invariant only under the symmetries of the solid under consideration, which restricts the number of possible symmetry groups to those whose invariant two-index tensors are accidentally isotropic. As we saw above, the cubic group has this property.

Moving away from the background, the constraints become more severe. The fluctuating degrees of freedom are the solid's phonons, which can be parametrized by a 3-vector field $\vec{\pi}(x)$, and the metric perturbations. To discuss possible anisotropies of scalar correlation functions, it is sufficient to focus on the phonons: the longitudinal one mixes with the scalar metric perturbations, and so any anisotropies in its dynamics will be reflected in scalar correlation functions. In particular, the two-point function is determined by the phonons' quadratic Lagrangian, which takes the general form

$$\mathcal{L}_2 = A_{ij}\dot{\pi}^i\dot{\pi}^j + B_{ijlm}\partial^i\pi^j\partial^l\pi^m \,, \tag{2.3}$$

where A^{ij} and B^{ijlm} are tensors that are invariant under the symmetry group of the solid. We see that for the scalar two-point function to be isotropic, one also needs the invariant four-index tensors to be isotropic. As we saw above, the cubic group does not pass this test.

We will show in Sect. 2.1 that the only discrete subgroup of rotations with the above properties is the icosahedral group—the symmetry group of an icosahedron. The natural question now is which observables are going to exhibit the anisotropies associated with such an icosahedron: there should be many preferred directions in

[12] See also [60–63] for more general applications of solids in astrophysics and cosmology.

the sky (20, 30, or 12, depending on whether one counts the faces, edges, or vertices), which should show up in correlation functions. The question can be approached once again in terms of invariant tensors. The scalar three-point function is determined by the cubic Lagrangian for the phonons, which now can involve a six-index tensor:

$$\mathcal{L}_3 \supset T^{ijklmn}\partial_i\pi_j\partial_k\pi_l\partial_m\pi_n \,. \tag{2.4}$$

We will show that the icosahedral group allows for the anisotropic invariant tensor

$$T^{ijklmn}_{\text{aniso}} \propto 2(\gamma+2)\,\delta^{ijklmn} + (\gamma+1)\left(\delta^{ijkl}\delta^{mn}\delta^{m\,i+1} + \cdots\right) + \left(\delta^{ijkl}\delta^{mn}\delta^{m\,i-1} + \cdots\right) \,,$$

where γ is the golden ratio, the six-index and four-index deltas are nonzero only if *all* their indices take the same value, and $i+1$ and $i-1$ are to be interpreted modulo 3, that is, $3+1 \to 1$ and $1-1 \to 3$. This makes the three-point function potentially highly anisotropic. In fact, we will show that there is a choice of Lagrangian coefficients for which the three-point function is *completely* anisotropic, in the sense that it has exactly vanishing overlap with any three-point function template associated with isotropic models.

Anisotropies can show up in the tensor spectrum as well. The reason is that the quadratic Lagrangian for tensor modes also involves a six-index invariant tensor,

$$\mathcal{L}_2^{(\gamma)} \supset C^{ijklmn}\,\partial_i\gamma_{jk}\,\partial_l\gamma_{mn} \,. \tag{2.5}$$

However, we will see that such six-index tensor can receive anisotropic contributions only from higher-derivative terms in the Lagrangian. Our hope is that it might be consistent within our effective theory to assume that these are so large as to yield order-one anisotropies in the tensor spectrum, and in the examples we considered in Sect. 3 we found no indication that this disrupts the technical naturalness of the effective theory. Notice that a strong anisotropy in the tensor spectrum can in principle reconcile tensions between a large tensor signal in a small patch of the sky and little or no signal in a whole-sky average, like the original—then evaporated—BICEP2/Planck tension.

Finally, we should emphasize that when we talk about 'solids' we do not mean systems with an underlying crystal structure, but rather continuous, homogeneous solid media, which can be more symmetric than crystals in the continuum limit. For instance, there is no crystal with icosahedral symmetry group, but it is perfectly consistent to assume such a symmetry for a continuous medium (in fact, there are *quasi*-crystals with icosahedral symmetry [64].) Even though the solids of everyday life are not homogeneous at microscopic scales, there is no *a priori* reason why there could not exist (perhaps strongly coupled) field theories that at finite density exhibit perfectly homogeneous solid-like states. If one is uncomfortable with such an assumption, one can regard our inflationary model simply as a system of three scalar fields with certain symmetries. As we reviewed in the introduction, the low-energy effective field theory is the same, which makes the difference between the two viewpoints unsubstantial.

2.1 Hunting for the right symmetry group

We now want to generalize all of the above to a more general solid, invariant only under a discrete subgroup of rotations, which nonetheless features the desiderata identified in the last section: an isotropic background stress-tensor, and an isotropic quadratic Lagrangian for the phonons. As we saw, at the mathematical level these requirements are equivalent to demanding that all invariant two-index and four-index tensors be fully isotropic for the symmetry group in question.

There is an infinite number of discrete subgroups of $SO(3)$, divided into two main classes: the crystallographic point groups and non-crystallographic ones. Let's start with the former class. A crystallographic point group is the symmetry group of a crystal system that can fill all of space. This means that the group has to map all the lattice points into one another, which is a stronger requirement than being simply a subgroup of rotations. Since there is only a finite number of crystal systems—triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal, and cubic—there is only a finite number of crystallographic point groups. Except for the hexagonal one, all crystal systems can be defined in terms of their three primitive lattice basis vector, let us call them $\vec{a}$, $\vec{b}$, and $\vec{c}$. Then, the two-index tensor

$$T^{ij} = a^i a^j + b^i b^j + c^i c^j \tag{2.6}$$

is invariant under the corresponding crystallographic point group. However, this tensor is not invariant under general $SO(3)$ rotations unless $\vec{a}$, $\vec{b}$, and $\vec{c}$ all have the same length and are all orthogonal to one another. So, only the cubic crystal survives. Still, as already pointed out in the last section, the cubic group fails our test at the four-index level, because the tensor structure

$$a^i a^j a^l a^m + b^i b^j b^l b^m + c^i c^j c^l c^m \tag{2.7}$$

is invariant under the cubic group but not under general $SO(3)$ rotations. We thus reach the conclusion that no crystallographic point group can meet both of our criteria [13].

The non-crystallographic point groups are the icosahedral group, the infinitely many C_n groups (n-fold rotations about a given axis), and the extensions of C_n that include some kind of reflection. In the two last cases, already at the two-index level we can easily construct invariant tensors that are not $SO(3)$ invariant: for instance, the projector onto the plane perpendicular to the rotation axis.

So, all our bets are on the icosahedral group—the symmetry group of the icosahedron. The icosahedron has 20 triangular faces, 30 edges, and 12 vertices, and there are 60 proper rotations that maps it into itself. Following [65], we orient our cartesian

[13] The only possible exception to this argument is the hexagonal crystal, some of whose links are not primitive lattice vectors but rather suitable linear combinations thereof. Still, one can easily show that certain two-index tensors that are invariant under the hexagonal group are not $SO(3)$ invariant, e.g.

$$T^{ij} \propto b_1^i b_1^j + b_2^i b_2^j + b_3^i b_3^j \,, \tag{2.8}$$

where $b_1 = (1,0,0)$, $b_2 = (1/2, \sqrt{3}/2, 0)$, $b_3 = (-1/2, \sqrt{3}/2, 0)$.

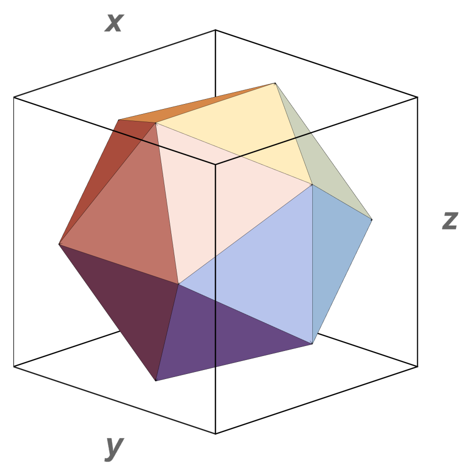


Figure 3: *The relative orientation between our coordinate system and the icosahedron discussed in the text.*

axes so that the icosahedron has two sides parallel to x, two parallel to y, and two parallel to z, as depicted in Fig. 3. In this case, the coordinates of the vertices are (up to an overall rescaling)

$$(\pm\gamma, \pm 1, 0)\,, \qquad (0, \pm\gamma, \pm 1)\,, \qquad (\pm 1, 0, \pm\gamma)\,, \tag{2.9}$$

where $\gamma = \frac{\sqrt{5}+1}{2}$ is the golden ratio.

We can find the invariant tensors in the following way. By definition, the two-index invariant tensors should satisfy

$$T^{ij} = T'^{ij} \equiv I^i{}_a\, I^j{}_b\, T^{ab} \tag{2.10}$$

for each rotation matrix I that belongs to the icosahedral group (we refer the reader to [65] for the explicit form of the rotation matrices). Given that there are 60 elements in the icosahedral group and $9 = 3 \times 3$ entries in T^{ij}, (2.10) can be interpreted as a system of 60×9 linear equations for the entries of T^{ij}. The solutions are [14]

$$T^{11} = T^{22} = T^{33}, \qquad \text{all other } T^{ij} = 0\,. \tag{2.11}$$

[14] We used Mathematica to solve this linear system and those that follow.

That is, all two-index tensors that are invariant under the icosahedral group have the form

$$T^{ij} \propto \delta^{ij}\,, \tag{2.12}$$

and are therefore fully isotropic.

We can apply the same logic to the four-index invariant tensors:

$$T^{ijlm} = T'^{ijlm} \equiv I^i{}_a\, I^j{}_b\, I^l{}_c\, I^m{}_d\, T^{abcd}\,. \tag{2.13}$$

In this case, we have 60×81 linear equations for the $81 = 3^4$ entries of T^{ijlm}, and the solutions are

$$\begin{aligned}
&T^{1122} = T^{1133} = T^{2211} = T^{2233} = T^{3311} = T^{3322}\,,\\
&T^{1212} = T^{1313} = T^{2121} = T^{2323} = T^{3131} = T^{3232}\,,\\
&T^{1221} = T^{1331} = T^{2112} = T^{2332} = T^{3113} = T^{3223}\,,\\
&T^{1111} = T^{2222} = T^{3333} = T^{1122} + T^{1212} + T^{1221}\,,\\
&\text{all other } T^{ijlm} = 0\,.
\end{aligned} \tag{2.14}$$

These conditions can be rewritten compactly using only Kronecker deltas, which shows that all four-index invariant tensors are fully isotropic as well:

$$T^{ijlm} = A\,\delta^{ij}\delta^{lm} + B\,\delta^{il}\delta^{jm} + C\,\delta^{im}\delta^{jl} \tag{2.15}$$

for arbitrary A, B, and C.

Finally, let us consider the six-index invariant tensors. Following the same logic, we now have $60 \times 3^6 = 60 \times 729$ equations

$$T^{ijklmn} = T'^{ijklmn} \equiv I^i{}_a\, I^j{}_b\, I^k{}_c\, I^l{}_d\, I^m{}_e\, I^n{}_f\, T^{abcdef}\,. \tag{2.16}$$

On top of the isotropic solutions, schematically of the form $T \sim \delta\delta\delta$, we find an anisotropic one:

$$T^{ijklmn}_{\text{aniso}} = 2(\gamma{+}2)\,\delta^{ijklmn} + (\gamma{+}1)\left(\delta^{ijkl}\delta^{mn}\delta^{m\,i+1} + \cdots\right) + \left(\delta^{ijkl}\delta^{mn}\delta^{m\,i-1} + \cdots\right) \tag{2.17}$$

where the dots stand for all other combinations of four and two indices out of six, the delta tensors with more than two indices are 1 only if *all* those indices take the same value, and $i+1$ and $i-1$ are to be interpreted modulo 3, that is $3+1=1$ and $1-1=3$.

It is worth mentioning that one could have derived the invariant tensors above in a perhaps more intuitive fashion, by using as building blocks the 12 vectors $\vec{v}_a$ ($a = 1, \ldots, 12$) that define the icosahedron's vertices. Clearly, by taking suitable tensor products and summing over all the vertices, e.g.

$$T^{i_1 \cdots i_n} = \sum_a v_a^{i_1} \cdots v_a^{i_n} , \tag{2.18}$$

one gets tensors that are invariant under the icosahedral group. However, it is not obvious that one can get *all* the invariant tensors in this way. Our brute-force analysis above settles the question (and the answer is 'yes', at least up to the six-index level, if one includes tensor products of lower-order tensors as well.)

In conclusion, the icosahedral group has exactly the properties that we are after: all its two-index and four-index invariant tensors are isotropic, whereas its six-index ones are not. And it is the only subgroup of $SO(3)$ with these properties. From now on we will thus focus on a variant of solid inflation with icosahedral symmetry, which we dub 'icosahedral inflation.' As already emphasized, the anisotropy of the six-index invariant tensors translates into an anisotropy of the scalar three-point function. Our goal now is to compute such a three-point function.

2.2 Cubic Lagrangian of scalar modes

An anisotropic invariant six-index tensor can induce anisotropies in the scalar three-point function through the trilinear phonon interaction

$$\mathcal{L}_3 \propto T_{\rm aniso}^{ijklmn} \partial_i \pi_j \partial_k \pi_l \partial_m \pi_n \, . \tag{2.19}$$

However, to figure out the most general structure of the cubic Lagrangian compatible with our symmetries requires some work. In the $SO(3)$-invariant version of solid inflation, this task was straightforward: the full solid's Lagrangian only depends on the three invariants (1.60), each of which can be expanded in perturbations about the background solution, up to any desired order. In icosahedral inflation, we face the problem of classifying the allowed invariants of B^{IJ}. Since B^{IJ} starts at zeroth order in $\vec{\pi}$,

$$B \sim 1 + \partial\pi + \partial\pi\partial\pi \,, \tag{2.20}$$

to expand the Lagrangian to any given order in $\vec{\pi}$—cubic, in our case—we need to consider all orders in B^{IJ}. However, at high orders, in principle we have to include more and more invariants,

$$T_{I_1 J_1 \cdots I_n J_n} \, B^{I_1 J_1} \cdots B^{I_n J_n} \,, \tag{2.21}$$

where T is a generic $2n$-index tensor with icosahedral symmetry. We are not aware of any simplifying property of the icosahedral group analogous to the $SO(3)$ statement that an arbitrary invariant of B^{IJ} can be written as a non-linear function of the three fundamental invariants (1.60). Clearly, the number of independent invariants cannot be more than the number of independent components of B^{IJ}—six—but using the individual components of B^{IJ} would make our computations messy and unreadable.

To get around this problem, we can work directly with the *fluctuation* of B^{IJ} about its background, but, as we will see below, we will have to be careful about the non-linearly realized symmetries [15]. In SFSG gauge, our building block up to cubic

[15]In the standard effective field theory of inflation [49], it is straightforward to write the action directly in terms of the metric perturbations in unitary gauge, by using for instance $\delta g^{00} \equiv g^{00} + 1$. As emphasized in [52], in solid inflation the analogous variable in unitary gauge would be $\delta g^{ij} = g^{ij} - \delta^{ij}/a^2$, but this, unlike the full g^{ij}, does not transform nicely under the residual time diffeomorphisms, because the background δ^{ij}/a^2 does not.

order is

$$\begin{aligned}
B^{ij} &= -\frac{1}{N^2}\left(\dot{\pi}^i - N^k\partial_k\phi^i\right)\left(\dot{\pi}^j - N^k\partial_k\phi^j\right) + h^{km}\partial_k\phi^i\partial_m\phi^j \\
&\simeq \frac{\delta^{ij}}{a^2} + \frac{1}{a^2}\left(\partial^i\pi^j + \partial^j\pi^i\right) + \frac{1}{a^2}\partial_k\pi^i\partial^k\pi^j - \left(\dot{\pi}^i - N^i\right)\left(\dot{\pi}^j - N^j\right) \\
&\quad + \left(\dot{\pi}^i - N^i\right)N^k\partial_k\pi^j + \left(\dot{\pi}^j - N^j\right)N^k\partial_k\pi^i + 2\delta N\left(\dot{\pi}^i - N^i\right)\left(\dot{\pi}^j - N^j\right) + \cdots \\
&\equiv \frac{1}{a^2}\left(\delta^{ij} + \pi^{ij}\right)
\end{aligned} \tag{2.22}$$

where we defined π^{ij} as the fluctuating part of B^{ij}, and we stopped differentiating between the internal $I, J, \ldots$ indices and the spacial $i, j, \ldots$ ones (the reason is that the background $\langle\phi^I\rangle = x^I$ breaks spatial rotations and internal ones down to the diagonal combination.) Were we to expand the full non-linear action up to cubic order, we would have

$$F(B^{ij}) = F_0 + \frac{\partial F}{\partial B^{ij}}\Big|_0 \frac{\pi^{ij}}{a^2} + \frac{1}{2!}\frac{\partial^2 F}{\partial B^{ij}B^{kl}}\Big|_0 \frac{\pi^{ij}\pi^{kl}}{a^4} + \frac{1}{3!}\frac{\partial^3 F}{\partial B^{ij}B^{kl}B^{mn}}\Big|_0 \frac{\pi^{ij}\pi^{kl}\pi^{mn}}{a^6} + \cdots, \tag{2.23}$$

where the subscript zeros mean 'evaluated on the background.' By the background Friedmann equations, the first derivative of F with respect to B^{ij} can be related to ϵ [52]:

$$\frac{\partial F}{\partial B^{ij}}\Big|_0 = \epsilon\,\tfrac{1}{3}a^2F_0\,\delta_{ij}\,. \tag{2.24}$$

This result was derived for the original solid inflation model assuming $SO(3)$ invariance, but in the Appendix we prove that it holds for our icosahedral inflation case as well. The higher-derivatives of F do not enter the Friedmann equations, and therefore cannot be related simply to other background quantities. As we will see below, they do obey constraints coming from the non-linearly realized symmetries, but for the moment we can just parametrize them as the most general icosahedral-invariant tensors with the right index-permutation symmetries ($i \leftrightarrow j$, $(ij) \leftrightarrow (kl)$, and so on). Since the factors of π^{ij} they are contracted with have precisely the same permutation symmetries, we can simply write

$$
\begin{aligned}
F(B^{ij}) = F_0 \cdot \big[1 + \tfrac{1}{3}\epsilon\,\pi^{ii} + \alpha_4\big(\delta_{ij}\delta_{kl} + \beta_1\delta_{ik}\delta_{jl}\big)\pi^{ij}\pi^{kl} \\
+ \alpha_6\big(\delta_{ij}\delta_{kl}\delta_{mn} + \beta_2\delta_{ij}\delta_{km}\delta_{ln} + \beta_3\delta_{ik}\delta_{jn}\delta_{lm} + \beta_4\,T^{ijklmn}_{\rm aniso}\big)\pi^{ij}\pi^{kl}\pi^{mn} + \cdots\big]\,,
\end{aligned}
\tag{2.25}
$$

where the α's and β's are generic dimensionless coefficients, with a weak time-dependence that can be neglected to lowest order in slow-roll.

If for the moment we ignore the metric perturbations δN and N^i, then π^{ij} is simply

$$
\pi^{ij} = \big(\partial^i\pi^j + \partial^j\pi^i\big) + \partial_k\pi^i\partial_k\pi^j - a^2\,\dot\pi^i\dot\pi^j\,, \tag{2.26}
$$

and isolating the different orders in $\vec\pi$ in the action above is immediate. At the quadratic level we get

$$
\begin{aligned}
\mathcal{L}_2 &= F_0 \cdot \big[-\tfrac{1}{3}\epsilon a^2\,\dot{\vec\pi}^{\,2} + \big(\tfrac{1}{3}\epsilon + 2\alpha_4\beta_1\big)\,(\partial_i\pi_j)^2 + 2\alpha_4(2+\beta_1)\,(\partial_i\pi^i)^2\big] \\
&\equiv -\tfrac{1}{3}\epsilon a^2 F_0 \cdot \left[\dot{\vec\pi}^{\,2} - c_T^2\,\frac{(\partial_i\pi_j)^2}{a^2} - (c_L^2 - c_T^2)\,\frac{(\partial_i\pi^i)^2}{a^2}\right]\,,
\end{aligned}
\tag{2.27}
$$

where

$$
c_T^2 = 1 + \frac{6\alpha_4\beta_1}{\epsilon}\,, \tag{2.28}
$$

$$
c_L^2 = 1 + \frac{12\alpha_4(1+\beta_1)}{\epsilon} \tag{2.29}
$$

are the transverse and longitudinal phonon speeds. For these to be between 0 and 1, we need both α_4 and $\alpha_4\beta_1$ to be small,

$$
\alpha_4, \alpha_4\beta_1 = \mathcal{O}(\epsilon)\,, \tag{2.30}
$$

in analogy with the $F_Y + F_Z = \mathcal{O}(\epsilon)\cdot F$ requirement of the original solid inflation case [52]. In fact, in the Appendix we prove that the two propagation speed are related by the same constraint as in solid inflation, (1.70), so that—as anticipated—up to quadratic order in perturbations our model is indistinguishable from solid inflation.

(2.30) implies that, to lowest order in slow-roll, only the second line in (2.25) contributes to the cubic Lagrangian:

$$\begin{aligned}\mathcal{L}_3 \;\simeq\;& \alpha_6 F_0 \cdot \Big[(8-\beta_3)\left(\partial_i\pi^i\right)^3 + 4\beta_2\left(\partial_i\pi^i\right)\left(\partial_j\pi^k\right)^2 + (4\beta_2+3\beta_3)\,\partial_i\pi^i\partial_j\pi^k\partial_k\pi^j \\ & + 6\beta_3\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i + 8\beta_4\, T^{ijklmn}_{\text{aniso}}\partial_i\pi^j\partial_k\pi^l\partial_m\pi^n\Big]\,. \end{aligned} \tag{2.31}$$

But we are not done yet. Ref. [66] argued that the approximate internal scale invariance (1.62) manifests itself on the structure of the phonon self-interactions in the following way: the cubic action expanded about a phonon background $\vec{\pi}_0$ cannot correct the quadratic action for the fluctuations if the background is isotropic, $\partial_i\pi^0_j \propto \delta_{ij}$. Applying this requirement to our cubic action yields two constraints on the β's,

$$72 + 28\beta_2 + 12\beta_3 + 48(\gamma+2)\beta_4 \;=\; 0\,, \tag{2.32}$$

$$12\beta_2 + 12\beta_3 + 24(\gamma+2)\beta_4 \;=\; 0\,, \tag{2.33}$$

which allow us to eliminate β_2 and β_4,

$$\beta_2 \;=\; 3\beta_3 - 18 \tag{2.34}$$

$$\beta_4 \;=\; \frac{9-2\beta_3}{\gamma+2}\,. \tag{2.35}$$

We are thus left with only two free coefficients, α_6 and β_3, which from now on we will simply call α and β. In conclusion, the cubic Lagrangian for icosahedral inflation reads

$$\begin{aligned}\mathcal{L}_3 \;=\;& \alpha F_0 \cdot \Big[\,(8-\beta)\left(\partial_i\pi^i\right)^3 + (12\beta-72)\,\partial_i\pi^i\left(\partial_j\pi^k\right)^2 + (15\beta-72)\,\partial_i\pi^i\partial_j\pi^k\partial_k\pi^j \\ & + 6\beta\,\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i + \frac{8(9-2\beta)}{\gamma+2}\,T^{ijklmn}_{\text{aniso}}\,\partial_i\pi_j\partial_k\pi_l\partial_m\pi_n\Big] \end{aligned} \tag{2.36}$$

Recall that in the original solid inflation model there was only one free coefficient at this order, F_Y, appearing as an overall factor in front the cubic Lagrangian (1.85)—all the relative coefficients of the different terms were completely fixed. If we set our anisotropic structure to zero by setting $\beta = 9/2$, we recover precisely those ratios, and we get

$$\alpha = -\frac{2}{243}\frac{F_Y}{F} \qquad\qquad (\beta = 9/2)\,. \tag{2.37}$$

On the other hand, we will show in the next section that the choice $\beta = 8$ characterizes the *completely anisotropic* case, in the sense that the resulting three-point function has exactly zero overlap with all those that one could get from isotropic models. The cubic Lagrangian in this case is

$$\begin{aligned} \mathcal{L}_3 \;\; &= \;\; 8\alpha F_0 \cdot \Big[3\,\partial_i\pi^i \left(\partial_j\pi^k\right)^2 + 6\,\partial_i\pi^i\partial_j\pi^k\partial_k\pi^j \qquad\qquad (2.38) \\ &\quad +6\,\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i - \frac{7}{\gamma+2}\,T^{ijklmn}_{\text{aniso}}\,\partial_i\pi_j\partial_k\pi_l\partial_m\pi_n \Big] \qquad (\beta = 8)\,. \end{aligned}$$

2.3 The size and shape of non-gaussianities

Like in the original case of solid inflation, and for the same reasons spelled out there [52], the leading trilinear interactions we need to consider to compute the scalar three-point function are the phonon self-interactions we wrote down above. That is, we can neglect interactions involving the metric perturbations. The computation of the three-point function parallels that in [52], with obvious modifications due the new tensor structures we have in the cubic Lagrangian. Neglecting the weak time-dependence of the scalar modes outside the horizon, the result is

$$\begin{aligned} \langle\zeta_1\zeta_2\zeta_3\rangle \;\; &\simeq \;\; (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\,\times \\ &\quad (-)\frac{9}{32}\frac{H^4}{M_{\rm p}^4}\cdot\frac{\alpha}{\epsilon^3 c_L^{12}}\cdot\frac{Q(\vec{k}_1,\vec{k}_2,\vec{k}_3)U(k_1,k_2,k_3)}{k_1^3k_2^3k_3^3}\,, \end{aligned} \tag{2.39}$$

where

$$
\begin{aligned}
Q(\vec{k}_1,\vec{k}_2,\vec{k}_3) = {} & (8-\beta)\,k_1k_2k_3 + 6\beta\frac{\big(\vec{k}_1\cdot\vec{k}_2\big)\big(\vec{k}_2\cdot\vec{k}_3\big)\big(\vec{k}_3\cdot\vec{k}_1\big)}{k_1k_2k_3} \\
& + (9\beta-48)\left(k_1\frac{\big(\vec{k}_2\cdot\vec{k}_3\big)^2}{k_2k_3} + k_2\frac{\big(\vec{k}_3\cdot\vec{k}_1\big)^2}{k_3k_1} + k_3\frac{\big(\vec{k}_1\cdot\vec{k}_2\big)^2}{k_1k_2}\right) \\
& + \frac{-16\beta+72}{\gamma+2}\frac{1}{k_1k_2k_3}\Big(2(\gamma+2)k_1^ik_1^ik_2^ik_2^ik_3^ik_3^i \\
& + (\gamma+1)\big(k_1^ik_1^ik_2^ik_2^ik_3^{i+1}k_3^{i+1} + k_1^ik_1^ik_2^{i+1}k_2^{i+1}k_3^ik_3^i + k_1^{i+1}k_1^{i+1}k_2^ik_2^ik_3^ik_3^i \\
& + 4k_1^ik_1^ik_2^ik_2^{i+1}k_3^ik_3^{i+1} + 4k_1^ik_1^{i+1}k_2^ik_2^ik_3^ik_3^{i+1} + 4k_1^ik_1^{i+1}k_2^ik_2^{i+1}k_3^ik_3^i\big) \\
& + \big(k_1^ik_1^ik_2^ik_2^ik_3^{i-1}k_3^{i-1} + k_1^ik_1^ik_2^{i-1}k_2^{i-1}k_3^ik_3^i + k_1^{i-1}k_1^{i-1}k_2^ik_2^ik_3^ik_3^i \\
& + 4k_1^ik_1^ik_2^ik_2^{i-1}k_3^ik_3^{i-1} + 4k_1^ik_1^{i-1}k_2^ik_2^ik_3^ik_3^{i-1} + 4k_1^ik_1^{i-1}k_2^ik_2^{i-1}k_3^ik_3^i\big)\Big)\,,
\end{aligned}
\tag{2.40}
$$

and

$$
\begin{aligned}
U(k_1,k_2,k_3) = {} & \frac{2}{k_1k_2k_3\,(k_1+k_2+k_3)^3}\Big\{3\big(k_1^6+k_2^6+k_3^6\big) + 20k_1^2k_2^2k_3^2 \\
& + 18\big(k_1^4k_2k_3 + k_1k_2^4k_3 + k_1k_2k_3^4\big) + 12\big(k_1^3k_2^3 + k_2^3k_3^3 + k_3^3k_1^3\big) \\
& 9\big(k_1^5k_2 + 5\,\mathrm{perms}\big) + 12\big(k_1^4k_2^2 + 5\,\mathrm{perms}\big) \\
& + 18\big(k_1^3k_2^3k_3 + 5\,\mathrm{perms}\big)\Big\}\,.
\end{aligned}
\tag{2.41}
$$

The overall delta function leaves us with only two independent momenta, say $\vec{k}_2$ and $\vec{k}_3$. Usually, because of isotropy, the absolute orientation of these two vectors does not matter, and one needs only three independent quantities to characterize the kinematical configuration: the magnitudes k_2 and k_3, and the relative angle θ. For us, because of our anisotropies, the absolute orientation matters, and so we have to keep all the six components of $\vec{k}_2$ and $\vec{k}_3$. This complicates the analysis considerably. In particular, we cannot use the standard techniques of [56].

A convenient parametrization of $\vec{k}_2$ and $\vec{k}_3$ is the following one. Define θ_2 and ϕ_2 as the standard polar and azimuthal angles of $\vec{k}_2$, but define θ_3 and ϕ_3 as the polar and azimuthal angles of $\vec{k}_3$ with respect to a primed coordinate system in which the

z' axis is along $\vec{k}_2$, and the x' axis lies in the plane defined by the z and z' axes (see fig. 4). The cartesian components of $\vec{k}_2$ and $\vec{k}_3$ thus are

$$\begin{aligned}
\vec{k}_2 &= k_2\left(\sin\theta_2\cos\phi_2,\ \sin\theta_2\sin\phi_2,\ \cos\theta_2\right), \qquad (2.42)\\
\vec{k}_3 &= k_3\left(\sin\theta_2\cos\phi_2\cos\theta_3+\cos\theta_2\cos\phi_2\sin\theta_3\cos\phi_3-\sin\phi_2\sin\theta_3\sin\phi_3,\right.\\
&\quad \sin\theta_2\sin\phi_2\cos\theta_3+\cos\theta_2\sin\phi_2\sin\theta_3\cos\phi_3+\cos\phi_2\sin\theta_3\sin\phi_3,\\
&\quad \left.\cos\theta_2\cos\theta_3-\sin\theta_2\sin\theta_3\cos\phi_3\right). \qquad (2.43)
\end{aligned}$$

The advantage of this parametrization is that θ_3 is the relative angle between $\vec{k}_2$ and $\vec{k}_3$, and so any dependence on θ_3 is perfectly consistent with isotropy. Anisotropies show up as a non-trivial dependence on ϕ_2, ϕ_3, and θ_2.

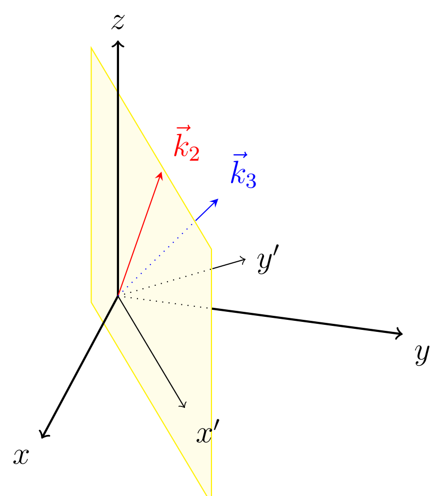


Figure 4: *The coordinate system defined in the text. The $\hat{z}$, $\hat{z}'=\hat{k}_2$, and $\hat{x}'$ axes all lie in the same plane.*

Following the standard conventions for correlation functions of the Newtonian potential Φ,

$$\begin{aligned}
\Phi &= \tfrac{3}{5}\zeta\,, \qquad (2.44)\\
\left\langle \Phi(\vec{k_1})\Phi(\vec{k_2})\right\rangle &= (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\frac{\Delta_\Phi}{k_1^3}\,, \qquad (2.45)\\
\left\langle \Phi(\vec{k_1})\Phi(\vec{k_2})\Phi(\vec{k_3})\right\rangle &= (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_2)f(\vec{k}_1,\vec{k}_2,\vec{k}_3)\,, \qquad (2.46)
\end{aligned}$$

we define $f_{\rm NL}$ using equilateral configurations:

$$f(\vec{k}_1,\vec{k}_2,\vec{k}_3)\big|_{\rm equil} = f_{\rm NL}\frac{6\Delta_\Phi^2}{k_1^6}\,. \tag{2.47}$$

However, the equilateral-triangle condition only fixes the relative angle θ_3, and so in our case the resulting $f_{\rm NL}$ depends non-trivially on the other angles, ϕ_2, ϕ_3, and θ_2. This is the fundamental difference between our $f_{\rm NL}$ and one defined in the Introduction, even though the two definitions look almost same. To get a readable expression, we average $f_{\rm NL}$ over ϕ_2 and ϕ_3,

$$\bar{f}_{\rm NL}(\theta_2) \equiv \frac{1}{4\pi^2}\int d\phi_2 d\phi_3\, f_{\rm NL}(\theta_2,\phi_2,\phi_3)\,. \tag{2.48}$$

The remaining dependence on θ_2 will still be a measure of anisotropy. For our three-point function, we get

$$\Delta_\Phi = \frac{9}{100}\frac{H^2}{M_{\rm p}^2}\cdot\frac{1}{\epsilon c_L^5}\,, \tag{2.49}$$

$$f(\vec{k}_1,\vec{k}_2,\vec{k}_3) = -\frac{15}{2}\frac{\alpha}{\epsilon c_L^2}\cdot\Delta_\Phi^2\cdot\frac{Q(\vec{k}_1,\vec{k}_2,\vec{k}_3)U(k_1,k_2,k_3)}{k_1^3k_2^3k_3^3}\,, \tag{2.50}$$

$$\bar{f}_{\rm NL}(\theta_2) = -\frac{\alpha}{\epsilon c_L^2}\left[\tfrac{19415}{378}(\beta-8)+\tfrac{104135}{6048}(2\beta-9)P_6(\cos\theta_2)\right], \tag{2.51}$$

where P_6 is the sixth-order Legendre polynomial.

The typical size of $f_{\rm NL}$ is the same as in the standard solid inflation case, parametrically as big as $1/\epsilon c_L^2$ if one assumes $\alpha\sim 1$ (analogous to $F_Y\sim F$ for solid inflation). But clearly the most interesting feature here is the angular dependence of $f_{\rm NL}$: the appearance of P_6 in $\bar{f}_{\rm NL}(\theta_2)$ is the first indication that the case with $\beta=8$ is a very special one, with a completely anisotropic $f_{\rm NL}$: if we average $\bar{f}_{\rm NL}$ over $\cos\theta_2$, which is equivalent to averaging the full $f_{\rm NL}$ over all angular variables, we get zero.

We can go further and, following [56], consider the overlap between our three-point function and other 'shapes.' This is defined as

$$\cos(f,f') \equiv \frac{f\cdot f'}{\sqrt{f\cdot f}\sqrt{f'\cdot f'}}\,, \tag{2.52}$$

where

$$f \cdot f' \equiv \sum_{\vec{k}_i} f\big(\vec{k}_1, \vec{k}_2, \vec{k}_3\big) f'\big(\vec{k}_1, \vec{k}_2, \vec{k}_3\big) / \big(\sigma_{k_1}^2 \sigma_{k_2}^2 \sigma_{k_3}^2\big)\,. \tag{2.53}$$

The sum runs over all triangles in momentum space, and is in fact an integral since the momenta are continuous variables.

If for f we take our shape (ignoring overall constant factors, which do not contribute to the overlap (2.52)),

$$f(\vec{k}_1, \vec{k}_2, \vec{k}_3) \to \frac{Q(\vec{k}_1, \vec{k}_2, \vec{k}_3) U(k_1, k_2, k_3)}{k_1^3 k_2^3 k_3^3}\,, \tag{2.54}$$

and for f' that coming from a general isotropic model,

$$f' \to f'(k_1, k_2, k_3)\,, \tag{2.55}$$

we find exactly vanishing overlap if $\beta = 8$. Again, the reason is manifest if, when computing the angular integrals for the overlap (2.52), we perform the integrals over ϕ_2 and ϕ_3 first:

$$\begin{aligned} \int d\!\cos\theta_2\, d\phi_2\, d\!\cos\theta_3\, d\phi_3\, f f' &= \int d\!\cos\theta_3 \frac{32\pi^2}{7} \frac{U\big(k_1, k_2, k_3\big)}{k_1^4 k_2^4 k_3^4} k_2^2 k_3^2\, f'(k_1, k_2, k_3) \\ \times \int d\!\cos\theta_2 \big((\beta - 8) G_1(k_2, k_3, \theta_2, \theta_3) &+ (2\beta - 9) G_2(k_2, k_3, \theta_2, \theta_3)\big)\,, \end{aligned} \tag{2.56}$$

where

$$G_1(k_2, k_3, \theta_2, \theta_3) = -\big[2(k_2^2 + k_3^3)\, P_2(\cos\theta_3) + k_2 k_3 \cos\theta_3 (1 + 3\cos^2\theta_3)\big]\,, \tag{2.57}$$

$$G_2(k_2, k_3, \theta_2, \theta_3) = \big[k_2^2\, P_2(\cos\theta_3) + 2 k_2 k_3\, P_3(\cos\theta_3) + k_3^2\, P_4(\cos\theta_3)\big] \cdot P_6(\cos\theta_2)\,, \tag{2.58}$$

and the P_n's are the Legendre polynomials. All quantities that only depend on the magnitudes k_1, k_2, k_3 factor out of the θ_2 integral, because they cannot depend on the orientation of the triangle defined by the momenta.

We clearly see the two limiting cases now. For $\beta = 9/2$, only the G_1 contribution survives, it has no θ_2 dependence, and we recover the results of the isotropic solid inflation case. On the other hand, for $\beta = 8$, G_1 is gone and G_2, being proportional to $P_6(\cos\theta_2)$, averages to zero when we integrate over θ_2.

3 Tensor modes with icosahedral symmetry

The existence of an anisotropic six-index invariant tensor suggests that anisotropies can also show up in the tensor modes' *two*-point function, because of the possible quadratic Lagrangian term

$$T_{\rm aniso}^{ijklmn}\,\partial_i\gamma_{jk}\,\partial_l\gamma_{mn}\,. \tag{3.1}$$

However, it is easy to convince oneself that such a term cannot arise from expanding the lowest-derivative action we have been working with so far,

$$S=\int d^4x\sqrt{-g}\left[\tfrac{1}{2}M_{\rm p}^2R+F(B^{IJ})\right], \tag{3.2}$$

simply because all possible anisotropies are in the structure of F, but its argument $B^{IJ}=g^{\mu\nu}\partial_\mu\phi^I\partial\phi^J$ does not involve derivatives of the metric.

On the other hand, in the presence of higher-derivative terms, one will generically get such a term. Consider for instance the invariant

$$(g\cdots g)^{\mu_1\nu_1\cdots\mu_6\nu_6}\cdot\nabla_{\mu_1}\nabla_{\nu_1}\phi^{I_1}\cdots\nabla_{\mu_6}\nabla_{\nu_6}\phi^{I_6}\cdot T_{\rm aniso}^{I_1\cdots I_6}\,, \tag{3.3}$$

where $(g\cdots g)$ stands schematically for any twelve-index tensor built out of the metric. Setting the ϕ^I's to their background values x^I, and expanding in powers of the tensor modes γ, the covariant derivatives $\nabla\nabla\phi^I$ have the schematic form

$$\nabla\nabla\phi^I\sim H+\partial\gamma\,; \tag{3.4}$$

and so, upon taking all the contractions in (3.3), one does expect to find the term (3.1) at quadratic order. Similar considerations apply to higher-derivative terms that involve higher powers of curvature tensors, for instance a trilinear term schematically of the form

$$(R^{\mu\nu\rho\sigma}\,\partial_\mu\phi^I\partial_\nu\phi^J\partial_\rho\phi^K\partial_\sigma\phi^L)^3\,, \tag{3.5}$$

with suitable contractions with our anisotropic invariant tensor $T_{\rm aniso}^{I_1\cdots I_6}$. (We need at least three Riemann tensors, because our $T_{\rm aniso}$ is totally symmetric, while $R^{\mu\nu\rho\sigma}$ has antisymmetry properties as well.)

However, if we want the anisotropic quadratic terms that we get from these higher-derivative corrections to compete with the purely isotropic ones we get from the Einstein-Hilbert action, we need to give the higher-derivative corrections a large coefficient, of order $M_{\rm p}^2/H^4$ in the examples above. This makes the smallness of F,

$$F \sim H^2 M_{\rm p}^2 \ll M_{\rm p}^4\,, \tag{3.6}$$

potentially unstable against quantum corrections. For instance, we expect graviton loops involving ϕ's on the external legs and the coupling (3.5) in the vertices, to drastically correct $F(B^{IJ})$. A quick order-of-magnitude estimate of this two-loop diagram

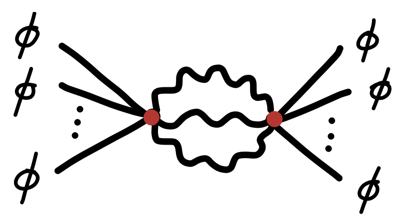


Figure 5: *Graviton loops with ϕ's on the external legs*

gives a correction to the effective action

$$\Delta F \sim \frac{M_{\rm p}^5}{H}\epsilon^{21/2}\,, \tag{3.7}$$

assuming the phonon speeds are relativistic, $c_L, c_T \sim 1$, and we cutoff the loop integrals at the solid's strong coupling scale, $\Lambda_{\rm strong} \sim \epsilon^{3/4}F^{1/4}$ [52]. For ΔF to be at most of order F, we need a small enough ϵ:

$$\epsilon \lesssim \big(H/M_{\rm p}\big)^{2/7}\,. \tag{3.8}$$

This goes in the opposite direction to the bound on ϵ that guarantees that perturbations are weakly coupled at freeze out [52],

$$\epsilon \gg \big(H/M_{\rm p}\big)^{2/3}\,, \tag{3.9}$$

but, for small $H/M_{\rm p}$, it is perfectly compatible with it.

Of course, this is not the end of story, and a systematic analysis of higher-derivative corrections will be done to check whether those terms are indeed allowed in Sect. 4. However, the above estimates suggest that it may be consistent to expect higher-derivative corrections to be large enough to yield order-one anisotropies in the tensor spectrum, but still small enough to preserve the technical naturalness of our effective field theory. At least, we can expect that there exist small anisotropic corrections to the tensor kinetic term. We will investigate the issue further in a later section. In this section, we will compute the small correction to the two-point function of the tensor modes due to small anisotropic corrections to the kinetic term, which we treat as small perturbations. The exact computation is also given in the Appendix.

3.1 A closer look at the icosahedral invariant tensor

In Sect. 2.1, we showed that the anisotropic icosahedral invariant six-index tensor is

$$T_{\rm aniso}^{ijklmn} = 2(\gamma+2)\,\delta^{ijklmn} + (\gamma+1)\left(\delta^{ijkl}\delta^{mn}\delta^{m\,i+1} + \cdots\right) + \left(\delta^{ijkl}\delta^{mn}\delta^{m\,i-1} + \cdots\right)\,. \tag{3.10}$$

We further showed that the icosahedral invariant scalar 3-point function of icosahedral inflation can be decomposed into an isotropic and a purely anisotropic part, in the sense that the overlap of the pure anisotropic part with any isotropic template vanishes. We used a Legendre polynomial expansion for that argument, but it turns out that one can directly decompose the tensor (3.10) itself, and, not surprisingly, the symmetry properties of the 3-point function just follow from such a decomposition.

First, notice that (3.10) is a totally symmetric tensor. A *generic* six-index spatial tensor can decomposed into irreps of $SO(3)$—from spin-0 to spin-6—as

$$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1} = 15\cdot\mathbf{0}\oplus 36\cdot\mathbf{1}\oplus 40\cdot\mathbf{2}\oplus 29\cdot\mathbf{3}\oplus 15\cdot\mathbf{4}\oplus 5\cdot\mathbf{5}\oplus 1\cdot\mathbf{6}\,. \tag{3.11}$$

However, upon totally symmetrizing, many such irreps are removed. Only one spin-

0, one spin-2, one spin-4, and one spin-6 are left. Moreover, since the tensor (3.10) is icosahedral invariant, so are its single trace and double trace. In Sect. 2.1, we also pointed out that all icosahedral invariant four-index and two-index tensors are isotropic. In other words, $T^{ijklmm}_{\rm aniso} \sim \delta\delta$ and $T^{ijkkmm}_{\rm aniso} \sim \delta$. Given that they are totally symmetric, they should be

$$T^{ijklmm}_{\rm aniso} = A\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right), \qquad T^{ijkkmm}_{\rm aniso} = 5A\,\delta^{ij}, \tag{3.12}$$

where A is an arbitrary constant. This implies that the single and double traces of (3.10) have only a spin-0 degree of freedom and nothing else. Therefore, we conclude that the tensor (3.10) has only one spin-0 and one spin-6 degrees of freedom.

$$T^{ijklmn}_{\rm aniso} = \mathbf{0} \oplus \mathbf{6}\,. \tag{3.13}$$

Its spin-0 dof is

$$T^{ijklmn}_{\rm aniso,\,0} = \frac{T^{iikkmm}_{\rm aniso}}{105} S^{ijklmn}\,, \tag{3.14}$$

where $S^{ijklmn} \equiv \left(\delta^{ij}\delta^{kl}\delta^{mn} + 14 \text{ other permutations}\right)$ [16] and the tensor (3.10) can be written as

$$T^{ijkkmm}_{\rm aniso} \equiv \frac{(\gamma+2)}{7} S^{ijklmn} + T^{ijklmn}_{\rm aniso,\,6}\,, \tag{3.15}$$

where $T^{ijklmn}_{\rm aniso,\,6}$ represents the spin-6 degree of freedom, which is purely anisotropic. Using this decomposition, we can reproduce the results of Sect. 2. In particular, if we plug (3.15) into the last term of the cubic Lagrangian, (2.38), which is purely anisotropic when we set $\beta = 8$, the terms involving $\frac{(\gamma+2)}{7} S^{ijklmn}$ in (3.15) cancel the first three terms of (2.38) exactly, leaving only the spin-6 trilinear interaction.

Since we used only the transformation properties of (3.10) under the rotation group, the argument we made in this subsection is very general, so not only the scalar three-point function, but any quantity involving (3.10) can be decomposed into spin-0 and spin-6 degrees of freedom.

[16] The presence of 105 is the analogue of that of 3 when a two-index tensor is decomposed into spin-0, 1 and 2 degrees of freedom: $T^{ij} = \frac{tr(T)}{3}\delta^{ij} + T^{[ij]} + \left(T^{(ij)} - \frac{tr(T)}{3}\delta^{ij}\right)$.

3.2 Icosahedral invariant two-point functions of the tensor modes

As we mentioned at the beginning of this section, we treat the icosahedral invariant spatial kinetic term of the tensor modes in a perturbative manner. In other words, we think of the icosahedral invariant spatial kinetic term as the perturbation of the non-interacting Lagrangian. Let us assume that we have the following Lagrangian:

$$\begin{aligned}\mathcal{L}_\gamma &= \frac{M_\mathrm{p}^2}{8}\int d^4x\, a^3\left[\dot{\gamma}_{ij}^2 - \frac{1}{a^2}\left(\partial_m\gamma_{ij}\right)^2 + \frac{\Delta c_\gamma^2}{a^2}T_{\mathrm{aniso}}^{ijklmn}\partial_i\gamma_{jk}\partial_l\gamma_{mn}\right]\,, \\ &\equiv \mathcal{L}_0 + \mathcal{L}_\mathrm{int}\,,\end{aligned} \tag{3.16}$$

where the free part is

$$\mathcal{L}_0 = \frac{M_\mathrm{p}^2}{8}\int d^4x\, a^3\left[\dot{\gamma}_{ij}^2 - \frac{1}{a^2}\left(\partial_m\gamma_{ij}\right)^2\right]\,, \tag{3.17}$$

the interaction part is

$$\mathcal{L}_\mathrm{int} = \frac{M_\mathrm{p}^2}{8}\int d^4x\, a^3\,\frac{\Delta c_\gamma^2}{a^2}T_{\mathrm{aniso}}^{ijklmn}\partial_i\gamma_{jk}\partial_l\gamma_{mn}\,, \tag{3.18}$$

and c_γ^2 is assumed to be small. Given the icosahedral invariant perturbation, the corrections to the two-point functions can be computed in a manner similar to the way in which we compute the three-point function of the scalar modes: the corrections can be found by computing the vacuum expectation value with the interacting Hamiltonian induced by the perturbation.

In Fourier space, our Lagrangian is

$$\begin{aligned}\mathcal{L}_\gamma \;=\; \frac{M_\mathrm{p}^2}{8}\int\frac{d\tau d^3k}{(2\pi)^3}\,a^2\Big[&\gamma'_{ij}(\vec{k})\gamma'_{ij}(-\vec{k}) - k^2\gamma_{ij}(\vec{k})\gamma_{ij}(-\vec{k}) \\ &+\Delta c_\gamma^2 k_i k_l T_{\mathrm{aniso}}^{ijklmn}\gamma_{jk}(\vec{k})\gamma_{mn}(-\vec{k})\Big]\,.\end{aligned} \tag{3.19}$$

The conventions we use are

$$\epsilon^s_{ij}\epsilon^{s'*}_{ij} = 2\delta^{ss'}\,, \tag{3.20}$$

$$\epsilon_{ii} = k_i\epsilon_{ij} = 0\,, \tag{3.21}$$

$$\gamma^s(\vec{k},\tau) = \gamma^s_{cl}(k,\tau)a^s(\vec{k}) + \gamma^s_{cl}(k,\tau)^* a^{s\dagger}(-\vec{k})\,, \tag{3.22}$$

$$\left[a^s(\vec{k}),\ a^{s'\dagger}(\vec{k}')\right] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta^{ss'}\,, \tag{3.23}$$

where the classical solution $\gamma^s_{cl}(k,\tau)$ obeys the equation of motion obtained by varying $\mathcal{L}_0$

$$\frac{d^2}{d\tau^2}\gamma_{cl} + 2aH\frac{d}{d\tau}\gamma_{cl} + k^2\gamma_{cl} = 0\,, \tag{3.24}$$

$$\gamma_{cl}(k,\tau) = \frac{1}{M_{\rm p}a}\frac{e^{-ik\tau}}{\sqrt{k}}\left(1-\frac{i}{k\tau}\right)\,. \tag{3.25}$$

3.2.1 Corrections to $\langle\gamma^+\gamma^+\rangle$ and $\langle\gamma^-\gamma^-\rangle$

As we proposed, let us take a similar approach to computing the three-point function:

$$\delta\langle\gamma^+(\tau)^2\rangle = -i\int_{-\infty}^{\tau} d\tau'\ \left\langle\Omega(-\infty)\right|\left[\gamma^+(\tau)^2, H_{\rm int}(\tau')\right]\left|\Omega(-\infty)\right\rangle\,, \tag{3.26}$$

where $H_{\rm int} = -\int d^3x\,\mathcal{L}_{\rm int}$. To be more explicit, defining $\gamma^s_i \equiv \gamma^s(\vec{k}_i,\tau)$,

$$\begin{aligned}\delta\left\langle\gamma^+_1\gamma^+_2\right\rangle \;=\; & \frac{iM_{\rm p}^2\Delta c_\gamma^2}{8}\int_{-\infty}^{\tau}\frac{d\tau' d^3p_1 d^3p_2}{(2\pi)^3}\,\delta^3(\vec{p}_1+\vec{p}_2)\,a^2 T^{ijklmn}_{\rm aniso}p_{1i}p_{1l}\epsilon^+_{jk}(\vec{p}_1)\epsilon^+_{mn}(-\vec{p}_1)\\ & \times\left\langle\left[\gamma^+_1\gamma^+_2,\gamma^+(\vec{p}_1,\tau')\gamma^+(\vec{p}_2,\tau')\right]\right\rangle\,.\end{aligned} \tag{3.27}$$

The commutator in the second line is

$$\begin{aligned}\left\langle\left[\gamma^+_1\gamma^+_2,\gamma^+(\vec{p}_1,\tau')\gamma^+(\vec{p}_2,\tau')\right]\right\rangle = (2\pi)^6&\left(\delta^3(\vec{p}_1+\vec{k}_2)\delta^3(\vec{p}_2+\vec{k}_1) + \vec{k}_1\leftrightarrow\vec{k}_2\right)\\ \times&\left(\gamma_{\rm cl,1}\gamma_{\rm cl,2}\gamma_{\rm cl}(p_1,\tau')^*\gamma_{\rm cl}(p_2,\tau')^* - \gamma^*_{\rm cl,1}\gamma^*_{\rm cl,2}\gamma_{\rm cl}(p_1,\tau')\gamma_{\rm cl}(p_2,\tau')\right)\,.\end{aligned} \tag{3.28}$$

Then, (3.27) becomes

$$
\begin{aligned}
\delta\left\langle\gamma_1^+\gamma_2^+\right\rangle &= \frac{iM_{\rm p}^2\Delta c_\gamma^2}{8}\int_{-\infty}^{\tau} d\tau' d^3p_1 d^3p_2\, a^2\, T^{ijklmn}_{\rm aniso} p_{1i}p_{1l}\epsilon^+_{jk}(\vec{p}_1)\epsilon^+_{mn}(-\vec{p}_1) \\
&\quad \times \left(\gamma_{{\rm cl},1}\gamma_{{\rm cl},2}\gamma_{\rm cl}(p_1,\tau')^*\gamma_{\rm cl}(p_2,\tau')^* - \gamma^*_{{\rm cl},1}\gamma^*_{{\rm cl},2}\gamma_{\rm cl}(p_1,\tau')\gamma_{\rm cl}(p_2,\tau')\right) \\
&\quad \times (2\pi)^3\delta^3(\vec{p}_1+\vec{p}_2)\left(\delta^3(\vec{p}_1+\vec{k}_2)\delta^3(\vec{p}_2+\vec{k}_1)+\vec{k}_1\leftrightarrow\vec{k}_2\right) \\
&= \frac{iM_{\rm p}^2\Delta c_\gamma^2}{8}\int_{-\infty}^{\tau} d\tau' d^3p_1\, a^2\, T^{ijklmn}_{\rm aniso} p_{1i}p_{1l}\epsilon^+_{jk}(\vec{p}_1)\epsilon^+_{mn}(-\vec{p}_1) \\
&\quad \times \left(\gamma_{{\rm cl},1}\gamma_{{\rm cl},2}\gamma_{\rm cl}(p_1,\tau')^*\gamma_{\rm cl}(p_1,\tau')^* - \gamma^*_{{\rm cl},1}\gamma^*_{{\rm cl},2}\gamma_{\rm cl}(p_1,\tau')\gamma_{\rm cl}(p_1,\tau')\right) \\
&\quad \times (2\pi)^3\left(\delta^3(\vec{p}_1+\vec{k}_2)\delta^3(-\vec{p}_1+\vec{k}_1)+\vec{k}_1\leftrightarrow\vec{k}_2\right) \\
&= \frac{iM_{\rm p}^2\Delta c_\gamma^2}{4}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\, T^{ijklmn}_{\rm aniso} k_{1i}k_{1l}\epsilon^+_{jk}(\vec{k}_1)\epsilon^+_{mn}(-\vec{k}_1) \\
&\quad \times \int_{-\infty}^{\tau} d\tau'\, a^2\left(\gamma_{{\rm cl},1}\gamma_{{\rm cl},2}\gamma_{\rm cl}(k_1,\tau')^*\gamma_{\rm cl}(k_1,\tau')^* - c.c.\right) \\
&\equiv \frac{M_{\rm p}^2\Delta c_\gamma^2}{4}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\, T^{ijklmn}_{\rm aniso} k_{1i}k_{1l}\epsilon^+_{jk}(\vec{k}_1)\epsilon^+_{mn}(-\vec{k}_1)\times I(\tau;\,-\infty)\,,
\end{aligned}
\tag{3.29}
$$

where

$$
I(\tau_1;\,\tau_2) = J(\tau_1;\,\tau_2) + J^*(\tau_1;\,\tau_2)\,, \tag{3.30}
$$

$$
J(\tau_1;\,\tau_2) = i\gamma_{\rm cl}(k_1,\tau_1)\gamma_{\rm cl}(k_2,\tau_1)\int_{\tau_2}^{\tau_1} {\rm d}\tau'\, a^2\gamma_{\rm cl}(k_1,\tau_2)^*\gamma_{\rm cl}(k_2,\tau_2)^*\,. \tag{3.31}
$$

Substituting (3.25) into the above integrals, we obtain

$$
I(\tau;\,-\infty) = \frac{2}{M_{\rm p}^4 a^2}\frac{1}{2k_1^5\tau^2}\left(k_1^2\tau^2+3\right)\,. \tag{3.32}
$$

As a result, the correction to the two-point function at late times is

$$
\delta\left\langle\gamma_1^+\gamma_2^+\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)T^{ijklmn}_{\rm aniso}\hat{k}_{1i}\hat{k}_{1l}\epsilon^+_{jk}(\vec{k}_1)\epsilon^+_{mn}(-\vec{k}_1)\frac{H^2}{M_{\rm p}^2}\frac{3\Delta c_\gamma^2}{4k_1^3}\,. \tag{3.33}
$$

Similarly, $\delta\left\langle\gamma_1^-\gamma_2^-\right\rangle$ is

$$
\delta\left\langle\gamma_1^-\gamma_2^-\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)T^{ijklmn}_{\rm aniso}\hat{k}_{1i}\hat{k}_{1l}\epsilon^-_{jk}(\vec{k}_1)\epsilon^-_{mn}(-\vec{k}_1)\frac{H^2}{M_{\rm p}^2}\frac{3\Delta c_\gamma^2}{4k_1^3}\,. \tag{3.34}
$$

3.2.2 Corrections to $\langle\gamma^-\gamma^+\rangle$ and $\langle\gamma^+\gamma^-\rangle$

Using the same method, we get

$$\begin{aligned}
\delta\left\langle\gamma_1^-\gamma_2^+\right\rangle = \frac{iM_{\rm p}^2\Delta c_\gamma^2}{8}\int_{-\infty}^{\tau}\frac{d\tau' d^3p_1 d^3p_2}{(2\pi)^3}\,\delta^3(\vec{p}_1+\vec{p}_2)\,a^2\,T^{ijklmn}_{\rm aniso}p_{1i}p_{1l} \\
\times\Big(\epsilon^+_{jk}(\vec{p}_1)\epsilon^-_{mn}(-\vec{p}_1)\left\langle\left[\gamma_1^-\gamma_2^+,\gamma^+(\vec{p}_1,\tau')\gamma^-(\vec{p}_2,\tau')\right]\right\rangle \\
+\epsilon^-_{jk}(\vec{p}_1)\epsilon^+_{mn}(-\vec{p}_1)\left\langle\left[\gamma_1^-\gamma_2^+,\gamma^-(\vec{p}_1,\tau')\gamma^+(\vec{p}_2,\tau')\right]\right\rangle\Big)\,.
\end{aligned} \tag{3.35}$$

The two commutators above are

$$\begin{aligned}
\left\langle\left[\gamma_1^-\gamma_2^+,\gamma^+(\vec{p}_1,\tau')\gamma^-(\vec{p}_2,\tau')\right]\right\rangle = (2\pi)^6\delta^3(\vec{p}_1+\vec{k}_2)\delta^3(\vec{p}_2+\vec{k}_1) \\
\times\left(\gamma_{\rm cl,1}\gamma_{\rm cl,2}\gamma_{\rm cl}(p_1,\tau')^*\gamma_{\rm cl}(p_2,\tau')^*-\gamma^*_{\rm cl,1}\gamma^*_{\rm cl,2}\gamma_{\rm cl}(p_1,\tau')\gamma_{\rm cl}(p_2,\tau')\right)\,,
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
\left\langle\left[\gamma_1^-\gamma_2^+,\gamma^+(\vec{p}_1,\tau')\gamma^-(\vec{p}_2,\tau')\right]\right\rangle = (2\pi)^6\delta^3(\vec{p}_1+\vec{k}_1)\delta^3(\vec{p}_2+\vec{k}_2) \\
\times\left(\gamma_{\rm cl,1}\gamma_{\rm cl,2}\gamma_{\rm cl}(p_1,\tau')^*\gamma_{\rm cl}(p_2,\tau')^*-\gamma^*_{\rm cl,1}\gamma^*_{\rm cl,2}\gamma_{\rm cl}(p_1,\tau')\gamma_{\rm cl}(p_2,\tau')\right)\,.
\end{aligned} \tag{3.37}$$

Therefore,

$$\begin{aligned}
\delta\left\langle\gamma_1^-\gamma_2^+\right\rangle &= \frac{iM_\mathrm{p}^2\Delta c_\gamma^2}{8}\int_{-\infty}^{\tau} d\tau' d^3p_1 d^3p_2\, a^2\, T_\mathrm{aniso}^{ijklmn} p_{1i}p_{1l} \\
&\quad\times\left(\gamma_{\mathrm{cl},1}\gamma_{\mathrm{cl},2}\gamma_\mathrm{cl}(p_1,\tau')^*\gamma_\mathrm{cl}(p_2,\tau')^* - \gamma_{\mathrm{cl},1}^*\gamma_{\mathrm{cl},2}^*\gamma_\mathrm{cl}(p_1,\tau')\gamma_\mathrm{cl}(p_2,\tau')\right) \\
&\quad\times(2\pi)^3\delta^3(\vec{p}_1+\vec{p}_2)\Big(\epsilon_{jk}^+(\vec{p}_1)\epsilon_{mn}^-(-\vec{p}_1)\delta^3(\vec{p}_1+\vec{k}_2)\delta^3(\vec{p}_2+\vec{k}_1) \\
&\qquad\quad+\epsilon_{jk}^-(\vec{p}_1)\epsilon_{mn}^+(-\vec{p}_1)\delta^3(\vec{p}_1+\vec{k}_1)\delta^3(\vec{p}_2+\vec{k}_2)\Big) \\
&= \frac{iM_\mathrm{p}^2\Delta c_\gamma^2}{8}\int_{-\infty}^{\tau} d\tau' d^3p_1\, a^2\, T_\mathrm{aniso}^{ijklmn} p_{1i}p_{1l} \\
&\quad\times\left(\gamma_{\mathrm{cl},1}\gamma_{\mathrm{cl},2}\gamma_\mathrm{cl}(p_1,\tau')^*\gamma_\mathrm{cl}(p_1,\tau')^* - \gamma_{\mathrm{cl},1}^*\gamma_{\mathrm{cl},2}^*\gamma_\mathrm{cl}(p_1,\tau')\gamma_\mathrm{cl}(p_1,\tau')\right) \\
&\quad\times(2\pi)^3\Big(\epsilon_{jk}^+(\vec{p}_1)\epsilon_{mn}^-(-\vec{p}_1)\delta^3(\vec{p}_1+\vec{k}_2)\delta^3(-\vec{p}_1+\vec{k}_1) \\
&\qquad\quad+\epsilon_{jk}^-(\vec{p}_1)\epsilon_{mn}^+(-\vec{p}_1)\delta^3(\vec{p}_1+\vec{k}_1)\delta^3(-\vec{p}_1+\vec{k}_2)\Big) \\
&= \frac{iM_\mathrm{p}^2\Delta c_\gamma^2}{4}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\, T_\mathrm{aniso}^{ijklmn} k_{1i}k_{1l}\epsilon_{jk}^+(\vec{k}_1)\epsilon_{mn}^-(-\vec{k}_1) \\
&\quad\times\int_{-\infty}^{\tau} d\tau'\, a^2\left(\gamma_{\mathrm{cl},1}\gamma_{\mathrm{cl},2}\gamma_\mathrm{cl}(k_1,\tau')^*\gamma_\mathrm{cl}(k_1,\tau')^* - c.c.\right) \\
&\equiv \frac{M_\mathrm{p}^2\Delta c_\gamma^2}{4}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\, T_\mathrm{aniso}^{ijklmn} k_{1i}k_{1l}\epsilon_{jk}^+(\vec{k}_1)\epsilon_{mn}^+(-\vec{k}_1)\times I(\tau;\,-\infty)\,,
\end{aligned} \tag{3.38}$$

where $I(\tau;\,-\infty)$ is the same as before. As a result, the two-point function at late times is

$$\delta\left\langle\gamma_1^-\gamma_2^+\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)T_\mathrm{aniso}^{ijklmn}\hat{k}_{1i}\hat{k}_{1l}\epsilon_{jk}^+(\vec{k}_1)\epsilon_{mn}^-(-\vec{k}_1)\frac{H^2}{M_\mathrm{p}^2}\frac{3\Delta c_\gamma^2}{4k_1^3}\,. \tag{3.39}$$

Similarly, $\delta\left\langle\gamma_1^+\gamma_2^-\right\rangle$ is

$$\delta\left\langle\gamma_1^+\gamma_2^-\right\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)T_\mathrm{aniso}^{ijklmn}\hat{k}_{1i}\hat{k}_{1l}\epsilon_{jk}^-(\vec{k}_1)\epsilon_{mn}^+(-\vec{k}_1)\frac{H^2}{M_\mathrm{p}^2}\frac{3\Delta c_\gamma^2}{4k_1^3}\,. \tag{3.40}$$

3.2.3 A closer look at $T_\mathrm{aniso}^{ijklmn}\hat{k}_i\hat{k}_l\epsilon_{jk}^\pm(\vec{k})\epsilon_{mn}^\pm(-\vec{k})$

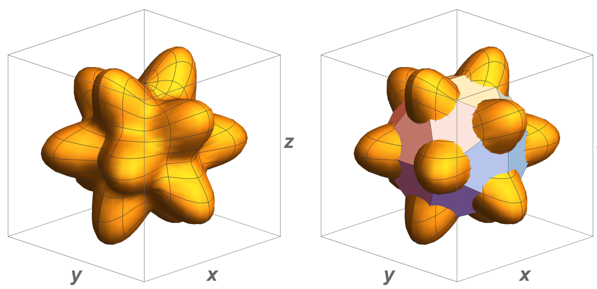
For convenience, let us define the following quantities:

$$\begin{aligned}
pp(\vec{k}) &\equiv T_\mathrm{aniso}^{ijklmn}\hat{k}_i\hat{k}_l\epsilon_{jk}^+(\vec{k})\epsilon_{mn}^+(-\vec{k})\,, \quad & mm(\vec{k}) &\equiv T_\mathrm{aniso}^{ijklmn}\hat{k}_i\hat{k}_l\epsilon_{jk}^-(\vec{k})\epsilon_{mn}^-(-\vec{k})\,, \\
pm(\vec{k}) &\equiv T_\mathrm{aniso}^{ijklmn}\hat{k}_i\hat{k}_l\epsilon_{jk}^+(\vec{k})\epsilon_{mn}^-(-\vec{k})\,, \quad & mp(\vec{k}) &\equiv T_\mathrm{aniso}^{ijklmn}\hat{k}_i\hat{k}_l\epsilon_{jk}^-(\vec{k})\epsilon_{mn}^+(-\vec{k})\,,
\end{aligned} \tag{3.41}$$

which appear in $\delta\langle\gamma^+\gamma^+\rangle$, $\delta\langle\gamma^-\gamma^-\rangle$, $\delta\langle\gamma^-\gamma^+\rangle$ and $\delta\langle\gamma^+\gamma^-\rangle$ correspondingly. Since $\epsilon^+_{mn}(-\vec{k}) = \epsilon^-_{mn}(\vec{k}) = \epsilon^+_{mn}(\vec{k})^*$, we have the following relations among them:

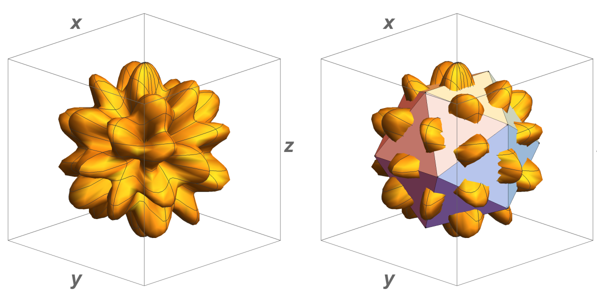
$$pp(\vec{k}) = mm(\vec{k})\,, \quad pm(\vec{k})^* = mp(\vec{k})\,. \tag{3.42}$$

The full mathematical expressions for these four quantities are very complicated, so we will not provide them in this thesis. Instead, we will present their plots. Since $pm(\vec{k})$ and $mp(\vec{k})$ are complex, we plot their absolute values.



(a) without icosahedron (b) with icosahedron

Figure 6: *$pp(\vec{k})$ and $mm(\vec{k})$*



(a) without icosahedron (b) with icosahedron

Figure 7: *$|pm(\vec{k})|$ and $|mp(\vec{k})|$*

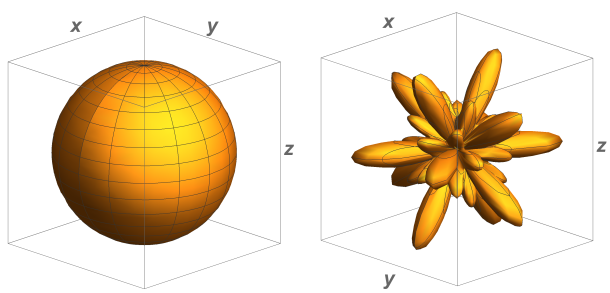
We can clearly observe that the spikes of $pp(\vec{k})$ and $mm(\vec{k})$ are aligned with the vertices of the underlying icosahedron and the spikes of $pm(\vec{k})$ and $mp(\vec{k})$ are aligned with the edges of the underlying icosahedron. Therefore, the geometric structure of the icosahedron still appears in the corrections to the two-point functions. Furthermore, as shown in Sect. 3.1, these spiky shapes decompose into a spin-0 sphere and spin-6 spikes. For instance, the spherical harmonic expansion of $pp(\vec{k})$ is

$$pp(\vec{k}) \equiv A_{00} Y_0^0(\theta,\, \phi) + \sum_{m=-6}^{6} A_{6m} Y_6^m(\theta, \phi)\,, \tag{3.43}$$

where

$$A_{00} = \frac{16(\gamma+2)}{7}\sqrt{\pi}, \quad A_{6\pm 6} = -5\gamma\sqrt{\frac{3\pi}{1001}}, \quad A_{6\pm 4} = 2(\gamma+2)\sqrt{\frac{\pi}{182}}\,,$$
$$A_{6\pm 2} = \gamma\sqrt{\frac{15\pi}{91}}, \quad A_{60} = -\frac{2(\gamma+2)}{7}\sqrt{\frac{\pi}{13}}\,. \tag{3.44}$$

The plots of these spin-0 and spin-6 parts are



(a) $A_{00} Y_0^0(\theta,\, \phi)$ (b) $\sum_{m=-6}^{6} A_{6m} Y_6^m(\theta, \phi)$

Figure 8: *Decomposition of* $pp(\vec{k})$

4 Higher-derivative corrections to solid inflation

4.1 Introduction of higher-derivative interactions

Although the analysis performed in [52] using the action $S = \int d^4x\,\sqrt{-g}F(X,Y,Z)$ is solid and self-consistent, it lacks of a thorough consideration of higher-derivative corrections. In most cases, large higher-derivative interactions can be ignored since their presence reduces the validity of a model. In this subsection, we check whether this is the case in solid inflation too, and if not, how it may affect the results of [52] (See [67] for a similar analysis in the context of supersolid.).

The full action, including the Einstein-Hilbert of the original solid inflation is

$$S = \int d^4x\sqrt{-g}\left\{\frac{M_{\rm p}^2}{2}R + F(X,Y,Z)\right\}\,. \tag{4.1}$$

On the background, both terms are of order $M_{\rm p}^2H^2$. Let us now introduce a higher-derivative term, for instance, $M^2F_1(X,Y,Z)R$ where R is the Ricci scalar and M is a constant with dimensions of mass. This non-minimal interaction can be comparable to the two terms in (4.1) if

$$M \sim M_{\rm p}\,. \tag{4.2}$$

We pay the price of having large higher-derivative interactions between phonons and gravitons: schematically $R \sim H^2 + \partial^2 h + \partial h\partial h + \mathcal{O}(h^3)$, so $M^2F_1(X,Y,Z)R$ can introduce $M_{\rm p}^2\partial^2h\partial\pi$ interactions. However, it might be possible to turn off such higher-derivative interactions with a proper tuning, as will be discussed later in this section.

Next, can we introduce higher-order terms in the curvature tensors? For instance, let us consider $cF_2(X,Y,Z)R^2$ where c is a c-number and $F_2(X,Y,Z) \sim 1$. In order for this term to be comparable with the Einstein-Hilbert and all other terms, we must have

$$c \sim \frac{M_{\rm p}^2}{H^2}\,, \tag{4.3}$$

so $cF_2(X,Y,Z)R^2$ introduces the following form of higher-derivative interactions of the gravitons:

$$M_{\mathrm{p}}^2 H^2 \partial^2 h \partial^2 h \,, \tag{4.4}$$

which lowers the cut-off scale to the Hubble scale. This spoils the loop correction argument we made in Sect. 3, since we can no longer use $\Lambda_{\text{strong}} \sim \epsilon^{3/4} F^{1/4}$. Moreover, these higher-derivative graviton interactions cannot be eliminated by tuning $cF_2(X,Y,Z)R^2$ alone unfortunately, even though in extreme cases it might be possible to turn off all such higher-derivative interactions by adjusting relative coefficients of all terms that are quadratic in the curvature tensors. Finding such fine-tunings is too challenging, so we will not consider it in this thesis.

Next let us turn our attention to higher-derivatives of the phonons. For instance, consider the following building block:

$$\begin{aligned} g^{\mu\nu} g^{\rho\sigma} \nabla_\mu \nabla_\nu \phi^I \nabla_\rho \nabla_\sigma \phi^J = \ddot{\pi}^I \ddot{\pi}^J - \frac{1}{a^2} \left(\ddot{\pi}^I \nabla^2 \pi^J + \nabla^2 \pi^I \ddot{\pi}^J \right) \\ + 3H \left(\ddot{\pi}^I \dot{\pi}^J + \dot{\pi}^I \ddot{\pi}^J \right) + 9H^2 \dot{\pi}^I \dot{\pi}^J \,. \end{aligned} \tag{4.5}$$

Again, we cannot eliminate the $\partial^2\pi\partial^2\pi$ terms by tuning the functions of this building block only, and at best, we might be able to do so with all possible higher-derivative building blocks. Again, we will not consider this possibility here.

In summary, the large higher-derivative interactions that are linear in the curvature tensors are the only viable candidates. In order to construct these interactions, let us introduce the new building blocks

$$C^{IJ} \equiv R^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J \,, \quad M^{IJKL} \equiv R^{\mu\nu\rho\sigma} \partial_\mu \phi^I \partial_\nu \phi^J \partial_\rho \phi^K \partial_\sigma \phi^L \,, \tag{4.6}$$

and our new action including the Einstein-Hilbert is

$$\begin{aligned} S = \int \mathrm{d}^4x \sqrt{-g} \left\{ \frac{M_{\mathrm{p}}^2}{2} R + F_B(B^{IJ}) + R F_R(B^{IJ}) + C^{IJ} F_C^{IJ}(B^{IJ}) \right. \\ \left. + M^{IJKL} F_M^{IJKL}(B^{IJ}) \right\} \,, \end{aligned} \tag{4.7}$$

where F_B, F_R, F_C^{IJ} and F_M^{IJKL} are generic functions of B^{IJ} [17].

4.2 The effective field theory of inflation formalism

The most general action, (4.7), is too complicated to analyze even on the background level. In order to overcome this difficulty, let us use the tools of the effective field theory of inflation [49]. In this formalism, inflation is described as a cosmological system where time diffeomorphisms are spontaneously broken by the homogeneous and isotropic background of the inflaton. In contrast, solid inflation can be described as a cosmological system where spatial diffeomorphisms are spontaneously broken but time diffeomorphisms are unbroken. In solid inflation, we can also choose 'Unitary Gauge,' in which our building blocks become

$$B^{IJ} \rightarrow g^{IJ}, \tag{4.8}$$

$$C^{IJ} \rightarrow R^{IJ}, \tag{4.9}$$

$$M^{IJKL} \rightarrow R^{IJKL}, \tag{4.10}$$

and our Lagrangian must be a function of these quantities.

However, there is one crucial difference between solid inflation and the effective field theory of inflation. In the latter, $\delta g^{00} = g^{00} + 1$ behaves as a scalar under the unbroken spatial diffeomorphisms, whereas in solid inflation, $g^{IJ} = \delta^{IJ}/a^2$ on the background, which is not invariant under the unbroken time diffomorphisms, making $\delta g^{IJ} = g^{IJ} - \delta^{IJ}/a^2$ transform non-trivially under the unbroken time diffeomorphisms. However, because g^{IJ}, R^{IJ} and R^{IJKL} are still scalars under time diffeomorphisms, let us try using these as our variables and see what happens. For the moment, let us consider only g^{IJ} for simplicity. Then our new Lagrangian can be written as a power series in g^{IJ},

$$F\big(g^{IJ}\big) = F_0 + F_1 g^{II} + F_2 g^{II} g^{JJ} + F_3 g^{IJ} g^{IJ} + \cdots, \tag{4.11}$$

[17] Of course, F_1 and F_2 are functions of $SO(3)$ invariant combinations of B^{IJ}, but for future convenience, let us consider them as the functions of B^{IJ}.

where $\cdots$ stands for higher-order terms in g^{IJ}. Indices of g^{IJ} should be contracted in a rotationally invariant way due to the combined rotational symmetry, and the coefficients are constants due to the combined translational symmetry. However, this way of writing the Lagrangian is problematic, since we need to have knowledge about arbitrary higher-order terms in g^{IJ} to analyze the spectra of perturbations order by order. This is why the effective field theory setup in [49] uses δg^{00} and $\delta K_{\mu\nu}$ as the parameters instead of g^{00} and $K_{\mu\nu}$; we thus have to rewrite (4.11) as a function of δg^{IJ}. Substituting $g^{IJ} = \delta^{IJ}/a^2 + \delta g^{IJ}$ into (4.11) and rewriting the Lagrangian as a power series in δg^{IJ},

$$F\big(g^{IJ}\big) = F_0'(t) + F_1'(t) g^{II} + F_2'(t)\delta g^{II}\delta g^{JJ} + F_3'(t)\delta g^{IJ}\delta g^{IJ} + \cdots \,. \tag{4.12}$$

Even though we started with constant coefficients in (4.11) due to the combined translational symmetry, our new coefficients must be functions of time because they are now combinations of the original coefficients and the scale factor $a(t)$ from δ^{IJ}/a^2. However, the time dependence of these new parameters is not arbitrary, but it must be constrained in such a way as to cancel the non-trivial transformation of δg^{IJ} under the unbroken time diffeomorphisms. Let us omit the prime on the new coefficients. Under $t \to t + \xi(t, \vec{x})$,

$$\delta g^{IJ} \to \delta g^{IJ} + \frac{2H}{a^2}\xi\delta^{IJ} + \frac{\dot{H} - 2H^2}{a^2}\xi^2\delta^{IJ} \cdots \,, \tag{4.13}$$

and

$$\begin{aligned} F\big(g^{IJ}\big) \quad \to \quad & F\big(g^{IJ}\big) + \xi\left(\dot{F}_0 + \frac{3}{a^2}\dot{F}_1\right) + \xi\delta g^{II}\left(\dot{F}_1 + \frac{4H}{a^2}(3F_2 + F_3)\right) \\ & + \xi^2\left(\frac{\ddot{F}_0}{2} + \frac{3}{2a^2}\ddot{F}_1 + \frac{12H^2}{a^4}(3F_2 + F_3)\right) + \cdots \,. \end{aligned} \tag{4.14}$$

In order to keep the Lagrangian invariant, we need the following constraints [18]:

$$\dot{F}_0 + \frac{3}{a^2}\dot{F}_1 = 0\,, \tag{4.15}$$

$$\dot{F}_1 + \frac{4H}{a^2}(3F_2 + F_3) = 0\,. \tag{4.16}$$

[18] The coefficient of ξ^2 can be automatically set to zero by having the two conditions above.

With these two constraints, the Lagrangian (4.12) is invariant under the full diffeomorphism group and can perfectly reproduce the analysis with (4.1).

Now let us generalize this methodology to (4.7). The full Lagrangian, up to quadratic order in fluctuations in unitary gauge, can be written as

$$
\begin{aligned}
\mathcal{L} = &\ F_{B0}(t) + a^2 F_{B1}(t) g^{II} + a^4 F_{B2}(t)\delta g^{II}\delta g^{JJ} + a^4 F_{B3}(t)\delta g^{IJ}\delta g^{IJ} \\
&+ \left(a^2 F_{C0}(t)\delta^{IJ} + a^4 F_{C1}(t) g^{IJ} + a^4 F_{C2}(t) g^{KK}\delta^{IJ} + a^6 F_{C3}(t)\delta g^{KK}\delta g^{IJ}\right. \\
&\left.+ a^6 F_{C4}(t)(\delta g^{KK})^2\delta^{IJ} + a^6 F_{C5}(t)\delta g^{IK}\delta g^{JK}\right) R^{IJ} \\
&+ \left(F_{R0}(t) + a^2 F_{R1}(t) g^{II} + a^4 F_{R2}(t)\delta g^{II}\delta g^{JJ} + a^4 F_{R3}(t)\delta g^{IJ}\delta g^{IJ}\right) R \\
&+ \left\{a^4 F_{M0}(t)\left(\delta^{IK}\delta^{JL} - \delta^{IL}\delta^{JK}\right)\right. \\
&+ a^6 F_{M1}(t)\left(\delta^{IK} g^{JL} - \delta^{IL} g^{JK} - \delta^{JK} g^{IL} + \delta^{JL} g^{IK}\right) \\
&+ a^6 F_{M2}(t) g^{MM}\left(\delta^{IK}\delta^{JL} - \delta^{IL}\delta^{JK}\right) + a^8 F_{M3}(t)\left(\delta g^{IK}\delta g^{JL} - \delta g^{IL}\delta g^{JK}\right) \\
&+ a^8 F_{M4}(t)\left(\delta^{IK}\delta g^{JM}\delta g^{ML} - \delta^{IL}\delta g^{JM}\delta g^{MK} - \delta^{JK}\delta g^{IM}\delta g^{ML} + \delta^{JL}\delta g^{IM}\delta g^{MK}\right) \\
&+ a^8 F_{M5}(t)\delta g^{MM}\left(\delta^{IK}\delta g^{JL} - \delta^{IL}\delta g^{JK} - \delta^{JK}\delta g^{IL} + \delta^{JL}\delta g^{IK}\right) \\
&\left.+ a^8 F_{M6}(t)\left(\delta g^{MM}\right)^2\left(\delta^{IK}\delta^{JL} - \delta^{IL}\delta^{JK}\right)\right\} R^{IJKL} + \cdots \\
\equiv &\ \mathcal{L}_B + \mathcal{L}_C + \mathcal{L}_R + \mathcal{L}_M\,,
\end{aligned}
\tag{4.17}
$$

where we define $\mathcal{L}_B$ to be the first line of the Lagrangian, $\mathcal{L}_C$ to be the next two lines, $\mathcal{L}_R$ to be the next one line and $\mathcal{L}_D$ to be the last six lines [19]. This Lagrangian will have the constraints:

$$0 = \dot{F}_{B0} + 3\dot{F}_{B1} + 6HF_{B1}\,, \tag{4.18a}$$

$$0 = \dot{F}_{C0} + \dot{F}_{C1} + 3\dot{F}_{C2} + 2H\left(F_{C0} + 2F_{C1} + 6F_{C2}\right)\,, \tag{4.18b}$$

$$0 = \dot{F}_{R0} + 3\dot{F}_{R1} + 6HF_{R1}\,, \tag{4.18c}$$

$$0 = \dot{F}_{M0} + 2\dot{F}_{M1} + 3\dot{F}_{M2} + 4H\left(F_{M0} + 3F_{M1} + \frac{9}{2}F_{M2}\right)\,, \tag{4.18d}$$

[19] The factors of the scale factor are introduced for a later convenience.

and

$$
\begin{aligned}
&\dot{F}_{B1} + 2H(F_{B1} + 6F_{B2} + 2F_{B3}) + (3H^2 + \dot{H})(\dot{F}_{C1} + 3\dot{F}_{C2}) \\
&+ 4H(3H^2 + \dot{H})(F_{C1} + 3F_{C2} + 3F_{C3} + 9F_{C4} + F_{C5}) \\
&+ 6(2H^2 + \dot{H})\dot{F}_{R1} + 24H(2H^2 + \dot{H})(3F_{R2} + F_{R3}) + 8H^2\left(\dot{F}_{M1} + \frac{3}{2}\dot{F}_{M2}\right) \\
&+ 16H^3\left(3F_{M1} + \frac{9}{2}F_{M2} + F_{M3} + 2F_{M4} + 6F_{M5} + 9F_{M6}\right) = 0\,.
\end{aligned}
\tag{4.18e}
$$

4.3 Slow-roll approximations

In [52], the authors identified the slow-roll approximations with an approximate symmetry under

$$\phi^I \to \lambda\phi^I\,, \tag{4.19}$$

which they called internal scale transformation. In this subsection, we will show that even in the presence of non-minimal interactions, the exact symmetry (4.19) corresponds to the de Sitter limit.

Consider an infinitesimal internal scale transformation

$$\phi^I \to \lambda\phi^I = (1+\omega)\phi^I \qquad \text{where } \omega \ll 1\,. \tag{4.20}$$

Under this transformation, g^{IJ} and R^{IJ} are rescaled by λ^2 because they have two ϕ's and R^{IJKL} is rescaled by λ^4 because it has four ϕ's [20]. Hence,

$$g^{IJ} \;\to\; \lambda^2 g^{IJ} = \frac{\delta^{IJ}}{a^2} + \frac{2\omega}{a^2}\delta^{IJ} + \delta g^{IJ} + \frac{\omega^2}{a^2}\delta^{IJ} + 2\omega\delta g^{IJ} + \cdots\,, \tag{4.21}$$

$$R^{IJ} \;\to\; \lambda^2 R^{IJ} = R^{IJ} + 2\omega R^{IJ} + \omega^2 R^{IJ}\,, \tag{4.22}$$

$$R^{IJKL} \;\to\; \lambda^4 R^{IJKL} = R^{IJKL} + 4\omega R^{IJKL} + 6\omega^2 R^{IJKL} + \cdots\,. \tag{4.23}$$

The invariance of the action under these transformations imposes further constraints on the coefficients in addition to what we already have due to invariance under time

[20] Notice that $\sqrt{-g}$ and R are not rescaled.

diffeomorphisms, which are

$$\begin{aligned} F_{B1} &= 3F_{B2} + F_{B3} = F_{C0} + 2F_{C1} + 6F_{C2} \\ &= 2F_{C2} + F_{C3} + 6F_{C4} = 2F_{C1} + 3F_{C3} + 2F_{C5} = F_{R1} = 3F_{R2} + F_{R3} \\ &= F_{M0} + 3F_{M1} + \frac{9}{2}F_{M2} = 3F_{M1} + F_{M3} + 2F_{M4} + 3F_{M5} \\ &= 3F_{M2} + 2F_{M5} + 6F_{M6} = 0\,. \end{aligned} \tag{4.24}$$

Now let us check whether these constraints result in a de Sitter background. The background density and pressure are

$$\begin{aligned} \rho =& -F_{B0} - 3F_{B1} + 6H^2\left(F_{C0} + 2F_{C1} + 6F_{C2}\right) - 6H^2\left(F_{R0} - 3F_{R1}\right) \\ &+ 12H^2\left(F_{M0} + 2F_{M1} + 3F_{M2}\right)\,, \end{aligned} \tag{4.25}$$

$$\begin{aligned} p =& F_{B0} + F_{B1} - 2(H^2 + 2\dot{H})F_{C0} - 4(H^2 + 2\dot{H})F_{C1} - 2H\dot{F}_{C1} - 12(H^2 + 2\dot{H})F_{C2} \\ &- 6H\dot{F}_{C2} + 2(3H^2 + 2\dot{H})F_{R0} - 6(5H^2 + 2\dot{H})F_{R1} - 12H\dot{F}_{R1} \\ &+ 4(H^2 - 2\dot{H})F_{M0} + 8(3H^2 - 2\dot{H})F_{M1} + 12(3H^2 - 2\dot{H})F_{M2}\,. \end{aligned} \tag{4.26}$$

Via the Friedmann equations, we get

$$fH^2 = -\frac{1}{3M_\mathrm{p}^2}\left(F_{B0} + 3F_{B1}\right)\,, \tag{4.27}$$

$$f\dot{H} = -\frac{1}{2M_\mathrm{p}^2}\left[-2F_{B1} + 4H^2\left(F_{C0} + 2F_{C1} + 6F_{C2}\right) - 2H\left(\dot{F}_{C1} + 3\dot{F}_{C2}\right)\right]\,, \tag{4.28}$$

where

$$f \equiv 1 - \frac{2}{M_\mathrm{p}^2}\left(F_{C0} + 2F_{C1} + 6F_{C2}\right) + \frac{2}{M_\mathrm{p}^2}\left(F_{R0} - 3F_{R1}\right) - \frac{4}{M_\mathrm{p}^2}\left(F_{M0} + 2F_{M1} + 3F_{M2}\right)\,. \tag{4.29}$$

Imposing (4.24) and (4.18) on H^2 and $\dot{H}$ gives

$$H^2\left(1 + \frac{2F_{R0}}{M_\mathrm{p}^2} - \frac{4F_{M0}}{3M_\mathrm{p}^2}\right) = -\frac{F_{B0}}{3M_\mathrm{p}^2}\,, \tag{4.30}$$

$$\dot{H}\left(1 + \frac{2F_{R0}}{M_\mathrm{p}^2} - \frac{4F_{M0}}{3M_\mathrm{p}^2}\right) = 0\,. \tag{4.31}$$

Therefore, as long as $1 + 2F_{R0}/M_\mathrm{p}^2 - 4F_{M0}/3M_\mathrm{p}^2 \neq 0$, we have $\dot{H} = 0$ and a finite H^2, which is constant. This result implies that if our system has the exact symmetry $\phi^I \to \lambda\phi^I$, it describes a de Sitter background.

4.4 Cosmological perturbations

In this subsection, we will perform the full perturbative analysis of scalar and tensor modes. In [52], it was shown that the spatially flat slicing gauge (SFSG) is the most convenient one for solid inflation, and we will work with this choice of gauge in a slightly different form:

$$\phi^i = x^i + \pi^i, \quad g_{00} = -(1+2\psi), \quad g_{0i} = g_{i0} = a(\partial_i B + B_i), \quad g_{ij} = a^2 \exp{(\gamma)}_{ij}\,, \tag{4.32}$$

where $\partial^i B_i = 0$, $\partial_i \gamma_{ij} = 0$ and $\gamma_{ii} = 0$ [21]. The phonon fields π^i can also be decomposed into scalar and vector components, and we follow the convention used in [52]:

$$\pi^i = \frac{\partial_i}{\sqrt{-\nabla^2}} \pi_L + \pi_T^i \qquad \text{where } \partial_i \pi_T^i = 0\,. \tag{4.33}$$

Even though we would have to work with the full action (4.17) to be completely thorough, we keep only $\mathcal{L}_C$ along with the Einstein-Hilbert and the original minimal interaction, $\mathcal{L}_B$, in order to prevent too much complication from diluting our message.

[21] ψ and $\partial_i B + B_i$ are the analogues of δN and N^i in ADM formalism.

4.4.1 Scalar modes

Each part of the action is given by the following:

$$\sqrt{-g}\mathcal{L}_{EH} = \frac{M_{\rm p}^2}{2}a^3\left(-9H^2\psi^2 + \frac{4H}{a}\partial_i\psi\partial^iB + 3H^2\partial_iB\partial^iB\right), \tag{4.34}$$

$$\begin{aligned}\sqrt{-g}\mathcal{L}_{B} = a^3\Big[&-\frac{1}{2}(F_{B0}+3F_{B1})\psi^2 + \frac{1}{2}(F_{B0}+F_{B1})\partial_iB\partial^iB\\
&+2F_{B1}\psi\partial_i\pi^i + 2aF_{B1}\partial_iB\dot{\pi}^i - a^2F_{B1}\dot{\vec{\pi}}^2\\
&+(F_{B1}+2F_{B3})(\partial_i\pi_j)^2 + 2(2F_{B2}+F_{B3})(\partial_i\pi^i)^2\Big],\end{aligned} \tag{4.35}$$

$$\begin{aligned}\sqrt{-g}\mathcal{L}_{C} = a^3\Big[&9H^2\left(F_{C0}+2F_{C1}+6F_{C2}\right)\psi^2 - 3(2H^2+\dot{H})F_{C0}\partial_iB\partial^iB\\
&-4(3H^2+\dot{H})\left(F_{C1}+3F_{C2}\right)\partial_iB\partial^iB + \frac{2H}{a}F_{C0}\partial_i\psi\partial^iB\\
&-4H^2\left(F_{C0}+2F_{C1}+6F_{C2}\right)\psi\partial_i\pi^i + 2H\left(\dot{F}_{C1}+3\dot{F}_{C2}\right)\psi\partial_i\pi^i\\
&+2H\left(3F_{C0}+4F_{C1}+12F_{C2}\right)\psi\partial_i\dot{\pi}^i\\
&-\frac{2}{a^2}\left(F_{C0}+2F_{C1}+4F_{C2}\right)\nabla^2\psi\partial_i\pi^i + 6a(H^2+\dot{H})F_{C0}\partial_iB\dot{\pi}^i\\
&+4a(3H^2+2\dot{H})\left(F_{C1}+3F_{C2}\right)\partial_iB\dot{\pi}^i + \frac{2}{a}\left(\dot{F}_{C1}+\dot{F}_{C2}\right)\nabla^2B\partial_i\pi^i\\
&-\frac{6H}{a}\left(F_{C0}+2F_{C1}+6F_{C2}\right)\nabla^2B\partial_i\pi^i\\
&+\frac{2}{a}\left(F_{C0}+2F_{C1}+4F_{C2}\right)\nabla^2B\partial_i\dot{\pi}^i\\
&-a^2\left(3(H^2+\dot{H})F_{C0}+2(3H^2+2\dot{H})F_{C1}+6(3H^2+2\dot{H})F_{C2}\right)\dot{\vec{\pi}}^2\\
&+(3H^2+\dot{H})\Big\{\left(F_{C0}+4F_{C1}+6F_{C2}+2F_{C5}\right)(\partial_i\pi_j)^2\\
&+\left(2F_{C1}+4F_{C2}+4F_{C3}+12F_{C4}+2F_{C5}\right)(\partial_i\pi^i)^2\Big\}\Big].\end{aligned} \tag{4.36}$$

Unlike in the original case, we cannot identify the coefficients in the Lagrangian with any physically meaningful quantities, such as H and $\dot{H}$. Of course, there is no logical flaw in leaving the coefficients in terms of F's. In that case, though, it would be hard to extract physical predictions from these coefficients because they are just some free parameters. To get around this issue, we further simplify our action by imposing exact internal scale invariance on $\mathcal{L}_C$ but not on $\mathcal{L}_B$ in order to make the predictions clear. Then, we have for $\mathcal{L}_C$

$$\sqrt{-g}\mathcal{L}_C = a^3\Bigg[- \dot{H}F_{C0}\partial_i B\partial^i B + \frac{2H}{a}F_{C0}\partial_i\psi\partial^i B + 2HF_{C0}\psi\partial_i\dot{\pi}^i + \frac{4}{a^2}F_{C2}\nabla^2\psi\partial_i\pi^i$$
$$+ 2a\dot{H}F_{C0}\partial_i B\dot{\pi}^i - \frac{4}{a}\dot{F}_{C2}\nabla^2 B\partial_i\pi^i - \frac{4}{a}F_{C2}\nabla^2 B\partial_i\dot{\pi}^i - a^2\dot{H}F_{C0}\dot{\vec{\pi}}^2$$
$$- (3H^2+\dot{H})F_{C3}\left\{3(\partial_i\pi^j)^2 + (\partial_i\pi^i)^2\right\}\Bigg]. \tag{4.37}$$

We have two issues with this version of $\mathcal{L}_C$: one is that the coefficient of $\dot{\pi}_i\dot{\pi}^i$ is suppressed by $\dot{H}$ but the coefficients of $(\partial\pi)^2$ are not, which means we might have superluminal phonon modes. Therefore, the coefficients of $(\partial\pi)^2$ should be suppressed as well; in other words, $F_{C3}=0$ to leading order in internal scale invariance. Recall that we had to have a similar tuning in the original solid inflation; $F_Y + F_Z \sim \epsilon F$, which means that this kind of tuning is not entirely new. The second issue is that as mentioned in Sect. 4.1, this large non-minimal interaction introduces $\partial^2 h\partial\pi$ interactions

$$-\frac{4}{a}F_{C2}\nabla^2 B\partial_i\dot{\pi}^i\,, \quad \frac{4}{a^2}F_{C2}\nabla^2\psi\partial_i\pi^i\,. \tag{4.38}$$

We can eliminate both of these by setting $F_{C2}=0$, which is the tuning mentioned in Sect. 4.1. After imposing these further tunings, $\mathcal{L}_C$ is now parameterized by only one free parameter, F_{C0}, which we define as $F_{C0}\equiv \alpha M_{\rm p}^2$, where α is a c-number:

$$\sqrt{-g}\mathcal{L}_C = a^3M_{\rm p}^2\alpha\Bigg[- \dot{H}\partial_i B\partial^i B + \frac{2H}{a}\partial_i\psi\partial^i B + 2H\psi\partial_i\dot{\pi}^i + 2a\dot{H}\partial_i B\dot{\pi}^i - a^2\dot{H}\dot{\vec{\pi}}^2\Bigg]. \tag{4.39}$$

Then, the full action is

$$\sqrt{-g}\mathcal{L} = a^3M_{\rm p}^2\Bigg[- \dot{H}\left(1+\alpha\right)\partial_i B\partial^i B - 3H^2\psi^2 + \frac{2H}{a}\left(1+\alpha\right)\partial_i\psi\partial^i B$$
$$+ 2a\dot{H}\left(1+\alpha\right)\partial_i B\dot{\pi}^i + 2\dot{H}\psi\partial_i\pi^i + 2H\alpha\psi\partial_i\dot{\pi}^i$$
$$- a^2\dot{H}M_{\rm p}^2\left(1+\alpha\right)\left\{\dot{\vec{\pi}}^2 - \frac{c_T^2}{a^2}(\partial_i\pi_j)^2 - \frac{c_L^2-c_T^2}{a^2}(\vec{\nabla}\cdot\vec{\pi})^2\right\}\Bigg], \tag{4.40}$$

where

$$c_T^2 = \frac{1}{1+\alpha}\left(1+\frac{2F_{B3}}{F_{B1}}\right) \qquad\qquad = \frac{1}{1+\alpha}c_{T,\mathrm{old}}^2\,, \tag{4.41}$$

$$c_L^2 = \frac{1}{1+\alpha}\left(1+\frac{4(F_{B2}+F_{B3})}{F_{B1}}\right) = \frac{1}{1+\alpha}c_{L,\mathrm{old}}^2\,, \tag{4.42}$$

and $c_{T,\mathrm{old}}^2$ and $c_{L,\mathrm{old}}^2$ are the speeds of sound in the original solid inflation model.

In (4.40), ψ and B are not dynamic, as expected, so we can integrate them out. The solutions in Fourier space are

$$B = \frac{a}{k}\frac{-(3a^2\dot{H}-\alpha k^2)\dot{\pi}_L/k^2+\dot{H}\pi_L/H}{(1+\alpha)-3a^2\dot{H}/k^2}\,, \tag{4.43}$$

$$\psi = -\frac{a^2\dot{H}}{kH}\frac{\dot{\pi}_L-\dot{H}\pi_L/H}{(1+\alpha)-3a^2\dot{H}/k^2}\,. \tag{4.44}$$

When α is zero, then ψ reduces to δN and B reduces to $\frac{a}{k}N_L$ from the introduction. If we plug these back into (4.40), the quadratic action is

$$S_2 = M_{\mathrm{p}}^2\int dt\,d^3k\,a^3\left[\frac{k^2/3}{1+k^2(1+\alpha)/3a^2H^2\epsilon}|\dot{\pi}_L-\dot{H}\pi_L/H|^2+\dot{H}(1+\alpha)c_L^2k^2|\pi_L|^2\right]\,. \tag{4.45}$$

Before moving on, let us first take a look at the curvature perturbation ζ. Since we introduced a new term to the matter Lagrangian, ζ also changes. In Fourier space, the new ζ is [22]

$$\zeta = \frac{1}{3}\left[-k\pi_L+\alpha\frac{H}{\dot{H}}\left(\frac{k^2}{a}B-k\dot{\pi}_L\right)\right] = -\frac{k}{3}\frac{(1+3\epsilon a^2H^2/k^2)\pi_L+\alpha H\dot{\pi}_L/\dot{H}}{(1+\alpha)+3\epsilon a^2H^2/k^2}\,. \tag{4.46}$$

This is definitely different from the original ζ in general. However, at the late times, which is the limit we are interested in, it reduces to the original ζ:

$$\zeta \to -\frac{k}{3}\frac{3\epsilon a^2H^2/k^2\pi_L}{3\epsilon a^2H^2/k^2} = -\frac{k}{3}\pi_L\,. \tag{4.47}$$

So, for the purpose of computing correlation functions, we can ignore any change to ζ due to the non-minimal interaction at the late times.

[22] See Appendix F for the proof.

Let us now get back to (4.45). Surprisingly, it looks very similar to the quadratic action in the original case. If we make the redefinitions

$$\vec{k}' \equiv \lambda \vec{k}\,, \tag{4.48}$$

$$M_{\rm p}'^2 \equiv \frac{M_{\rm p}^2}{\lambda^5}\,, \tag{4.49}$$

$$\pi_L'(k') \equiv \pi_L(k)\,, \tag{4.50}$$

where $\lambda \equiv (1+\alpha)^{1/2}$, then we can convert (4.45) into the following form:

$$S_2 = M_{\rm p}'^2 \int dt\, d^3k'\, a^3 \left[\frac{k'^2/3}{1 + k'^2/3a^2H^2\epsilon} |\dot{\pi}_L' - \dot{H}\pi_L'/H|^2 + \dot{H} c_{L,\rm old}^2 k'^2 |\pi_L'|^2 \right]\,, \tag{4.51}$$

which is exactly the same as the original solid inflation action. Therefore, the spectrum of π_L' should have the same form as (1.80). Using this and the fact that $\langle \pi\pi \rangle \propto \delta^3(k)/k^5$ in solid inflation [23], we can compute the spectrum of π_L with the action (4.45).

$$\begin{aligned} \langle \pi_L(k)\pi_L(k)\rangle &= \langle \pi_L'(\lambda k)\pi_L'(\lambda k)\rangle = \frac{1}{\lambda^8}\langle \pi_L'(k)\pi_L'(k)\rangle \\ &\propto \frac{1}{\lambda^8}\frac{H^2}{\epsilon c_{L,\rm old}^2 M_{\rm p}'^2} = \frac{1}{\lambda^3}\frac{H^2}{\epsilon c_{L,\rm old}^2 M_{\rm p}^2} = \frac{1}{(1+\alpha)^{3/2}}\frac{H^2}{\epsilon c_{L,\rm old}^2 M_{\rm p}^2}\,. \end{aligned} \tag{4.52}$$

Here, we still keep $c_{L,\rm old}$ since this is numerically equal to the original speed of sound. Therefore, compared to the original solid inflation model, the new scalar power spectrum retains a similar structure but gets enhanced by the factor of $\frac{1}{(1+\alpha)^{3/2}}$:

$$\langle \zeta\zeta \rangle = \frac{1}{(1+\alpha)^{3/2}} \langle \zeta\zeta \rangle_{\rm original}\,. \tag{4.53}$$

4.4.2 Tensor modes

Our tensor action is

$$S_\gamma = \int d^4x\, a^3 \left\{ \frac{M_{\rm p}^2}{2}\left[\frac{1}{4}\dot{\gamma}_{ij}^2 - \frac{1+\alpha}{4a^2}(\partial_m \gamma_{ij})^2 \right] + a^3 \left(\frac{F_{B1}}{2} + F_{B3} \right) (\gamma_{ij})^2 \right\}\,. \tag{4.54}$$

[23] $\langle \zeta\zeta \rangle \propto \delta^3(k)/k^3$ and $\zeta = -\frac{k\pi_L}{3}$.

Ignoring the mass term, which vanishes at the leading order in the slow-roll limit, the tensor mode gets enhanced by

$$\langle\gamma\gamma\rangle \propto \frac{1}{(1+\alpha)^{3/2}}\frac{H^2}{M_{\rm p}^2}\,. \tag{4.55}$$

Notice that the extra factor is the same as the enhancement of the scalar mode, $\frac{1}{(1+\alpha)^{3/2}}$. Therefore, despite the existence of the non-minimal interaction and the enhancement of each mode, the tensor-to-scalar ratio remains the same.

5 Concluding remarks

In Sect. 2, we showed that among the generalizations of solid inflation with discrete rotational symmetries, only the one with icosahedral symmetry is naturally compatible with the observed isotropy of the background and the scalar spectrum.

The associated scalar three-point function is in general highly anisotropic, and this suppresses its overlap with all the standard templates used in the CMB data analyses. For a specific choice of the Lagrangian coefficients ($\beta = 8$, in our notation), it is *completely* anisotropic, in the sense that such an overlap vanishes exactly. This leaves open the possibility that a large non-gaussian signal is hiding in the data, waiting to be unveiled by a dedicated anisotropic analysis.

It is worth pointing out that our anisotropies are not of the same nature as those discussed in [68, 69]: there, for any given realization one expects small anisotropies in the scalar spectrum; but, statistically speaking, these average to zero. On the other hand, in our case it is the statistical correlation functions themselves that are intrinsically anisotropic, potentially maximally so.

For the scalar modes, we see no reason for the $\beta = 8$ case to be preferred over others; for instance, we see no symmetry protecting it against quantum corrections. However, it is a simple, consistent limit of our theory, and we find it interesting that such a completely anisotropic limit exists at all. Is it an accidental feature of our truncation of the theory at the cubic/three-point function level, or does it survive at higher-orders as well?

Similar considerations apply to the tensor spectrum as well: in the presence of sizable higher-derivative corrections, it can be highly anisotropic, which makes the standard detection strategies inefficient, and calls for a dedicated analysis. However, as shown in Sect. 4, such sizable higher-derivative corrections disrupt the technical naturalness of the effective field theory. Therefore, in Sect. 3, we computed the anisotropic corrections to the tensor spectrum by treating the anisotropic kinetic term as a small perturbation.

For this purpose, we used a similar approach to computing the three-point function and found the corrections to the tensor power spectrum. In particular, the corrections to $\langle\gamma^-\gamma^+\rangle$ and $\langle\gamma^+\gamma^-\rangle$ are interesting because they are identically zero in all isotropic models. However, in our model, due to the inherent anisotropic structure, there can be mixing between $+$ and $-$ helicity modes, and this is why we have non-zero mixed spectra $\langle\gamma^-\gamma^+\rangle$ and $\langle\gamma^+\gamma^-\rangle$. If we find any non-zero signals of mixing between $+$ and $-$ modes in future observations, that would be a smoking gun signal of our anisotropic structure.

The complete analysis of higher-derivative interactions was performed in Sect. 4. We found that, at best, we could have large non-minimal interactions of a solid with gravity that are linear in the curvature tensors. In the presence of a particular type of the non-minimal interaction, $\mathcal{L}_C$, both the scalar and tensor power spectra get enhanced. However, since the enhancement factors for the scalar and tensor modes are equal, the tensor-to-scalar ratio remains the same, so we cannot find an observable signal for this large non-minimal interaction. This story could change if we consider other observables, such as tilts or three-point functions, which may be an interesting subject for future study.

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Appendix A Variation of the slow-roll parameters

In the literature, a few different conventions are used for the slow-roll parameters. In this Appendix, we provide those definitions and the relations between them.

- Defined by the Hubble constant:

$$\epsilon = -\frac{\dot{H}}{H^2}, \tag{A.1}$$

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = \frac{\ddot{H}}{H\dot{H}} - \frac{2\dot{H}}{H^2}. \tag{A.2}$$

- Defined by the background value of ϕ:

$$\epsilon_\phi = \frac{3}{2}\frac{\dot{\bar{\phi}}^2}{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})}, \tag{A.3}$$

$$\eta_\phi = -\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}H}. \tag{A.4}$$

- Defined by a potential $V(\phi)$:

$$\epsilon_V = \frac{M_{\rm p}^2}{2}\left(\frac{V'(\bar{\phi})}{V(\bar{\phi})}\right)^2, \tag{A.5}$$

$$\eta_V = M_{\rm p}^2\frac{V''(\bar{\phi})}{V(\bar{\phi})}. \tag{A.6}$$

A.1 Parameter ϵ

Using (1.15) and (1.16), it can be shown that

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\dot{\bar{\phi}}^2}{\frac{1}{2}\dot{\bar{\phi}}^2 + V} = \epsilon_\phi. \tag{A.7}$$

Hence ϵ and ϵ_ϕ are exactly the same without any approximations. To relate them to ϵ_V, let us take the slow-roll limit of (1.15) and (1.16) [24].

$$H^2 \simeq \frac{1}{3M_{\rm p}^2}V(\bar{\phi}), \tag{A.8}$$

$$\dot{H} \simeq -\frac{1}{2M_{\rm p}^2}\frac{V'(\bar{\phi})^2}{9H^2} \simeq -\frac{V'(\bar{\phi})^2}{6V(\bar{\phi})}, \tag{A.9}$$

[24] The symbol $\simeq$ means both sides are equal in the slow-roll limit.

where we used (1.14) in the first equality of $\dot{H}$. Plugging these two expressions into ϵ and ϵ_ϕ gives

$$\epsilon = \epsilon_\phi = -\frac{\dot{H}}{H^2} \simeq \frac{M_{\rm p}^2}{2}\left(\frac{V'(\bar{\phi})}{V(\bar{\phi})}\right)^2 = \epsilon_V \,. \tag{A.10}$$

In summary,

$$\epsilon = \epsilon_\phi \simeq \epsilon_V \,. \tag{A.11}$$

A.2 Parameter η

To get $\ddot{H}$, let us take the time derivative of (1.16):

$$\ddot{H} = -\frac{1}{M_{\rm p}^2}\ddot{\bar{\phi}}\dot{\bar{\phi}} \,. \tag{A.12}$$

Plugging this and (1.16) into η gives

$$\eta = \frac{\ddot{H}}{H\dot{H}} - \frac{2\dot{H}}{H^2} = \frac{2\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} - \frac{2\dot{H}}{H^2} = -2\eta_\phi + 2\epsilon \,. \tag{A.13}$$

Again, no approximation has been made. Hence we have $\eta = -2\eta_\phi + \epsilon$. To relate η_ϕ to η_V, let us take the time derivative of (1.14):

$$3\dot{H}\dot{\bar{\phi}} + 3H\ddot{\bar{\phi}} + V''(\bar{\phi})\dot{\bar{\phi}} \simeq 0 \,. \tag{A.14}$$

We thus have

$$\ddot{\bar{\phi}} \simeq -\frac{V''(\bar{\phi})\dot{\bar{\phi}}}{3H} - \frac{\dot{\bar{\phi}}\dot{H}}{H} \,. \tag{A.15}$$

Plugging this into η_ϕ gives

$$\eta_\phi \simeq \frac{V''(\bar{\phi})}{3H^2} + \frac{\dot{H}}{H^2} \simeq M_{\rm p}^2\frac{V''(\bar{\phi})}{V(\bar{\phi})} - \epsilon = \eta_V - \epsilon \,. \tag{A.16}$$

In summary,

$$\eta = -2\eta_\phi + 2\epsilon \simeq -2\eta_V + 4\epsilon \,. \tag{A.17}$$

Appendix B Tilts in single-field inflation

In (1.36) and (1.37), we defined the scalar and tensor tilts as follows:

$$n_s - 1 \equiv \frac{d\ln P_s(k)}{d\ln k}\,, \tag{B.1}$$

$$n_\gamma - 1 \equiv \frac{d\ln P_\gamma(k)}{d\ln k}\,. \tag{B.2}$$

In this Appendix, we would like to prove that $n_s - 1$ and $n_\gamma - 1$ can be expressed as in Sect. 1, because our tilts take a slightly different form from the literature.

As we mentioned in the introduction, the k dependence is hidden. Hence, we have to change a derivative variable from $\ln k$ to something else on which the quantities in the power spectra depend explicitly. The most straightforward option is t_0, the time at horizon crossing, $k = aH$, where every quantity is evaluated at t_0. Using the chain rule, we obtain

$$\frac{d}{d\ln k} = k\frac{d}{dk} = k\frac{dt_0}{dk}\frac{d}{dt_0} = aH_0\frac{1}{a(H_0^2 + \dot{H}_0)}\frac{d}{dt_0} = \frac{1}{H_0(1-\epsilon_0)}\frac{d}{dt_0}\,. \tag{B.3}$$

Then the tilts can be computed explicitly as follows:

$$\begin{aligned} n_s - 1 \equiv \frac{d\ln P_s(k)}{d\ln k} &= \frac{d\ln\left(H_0^2/\epsilon_0\right)}{d\ln k} = 2\frac{d\ln H_0}{d\ln k} - \frac{d\ln \epsilon_0}{d\ln k} \\ &= \frac{1}{H_0(1-\epsilon_0)}\left(\frac{2\dot{H}_0}{H_0} - \frac{\dot{\epsilon}_0}{\epsilon_0}\right) \simeq -2\epsilon_0 - \eta_0 \end{aligned} \tag{B.4}$$

and

$$n_\gamma - 1 \equiv \frac{d\ln P_\gamma(k)}{d\ln k} = \frac{d\ln H_0^2}{d\ln k} = 2\frac{d\ln H_0}{d\ln k} = \frac{1}{H_0(1-\epsilon_0)}\left(\frac{2\dot{H}_0}{H_0}\right) \simeq -2\epsilon_0\,. \tag{B.5}$$

Appendix C Minimally coupled Lagrangian with icosahedral symmetry

C.1 Background stress-energy tensor

To relate the background stress-energy tensor to F and its derivatives, it is useful to organize F's dependence on B^{IJ} in terms of the variables

$$X \equiv [B] \,, \qquad b^{IJ} \equiv \frac{B^{IJ}}{[B]} \qquad\qquad (B^{IJ} = g^{\mu\nu}\, \partial_\mu \phi^I \partial_\nu \phi^J) \,. \tag{C.1}$$

X depends on the overall normalization of B^{IJ} whereas b^{IJ} does not. As a result, the approximate internal scale invariance (1.62) translates into a weak X-dependence of F.

Taking the variation with respect to the metric for the solid action

$$S_{\text{solid}} = \int d^4x \sqrt{-g}\, F(X, b^{IJ}) \,, \tag{C.2}$$

we find the stress-energy tensor

$$T_{\mu\nu} = g_{\mu\nu} F - 2F_X \partial_\mu \phi^I \partial_\nu \phi^I - \frac{2}{X} F_{IJ} \big(\partial_\mu \phi^I \partial_\nu \phi^J - b^{IJ} \partial_\mu \phi^K \partial_\nu \phi^K \big) \,, \tag{C.3}$$

where the subscript X and IJ stand for partial derivatives w.r.t. X and b^{IJ}.

When we evaluate $T_{\mu\nu}$ on the background configuration, we can use the fact that F_{IJ} must be icosahedral invariant. As we saw, for a two-index tensor this implies that it is proportional to δ_{IJ}. The terms in parentheses in (C.3) thus cancel against each other, and we are left with the same background stress-energy tensor as in $SO(3)$-invariant solid inflation. In particular:

$$\rho = -F, \quad p = F - \tfrac{2}{a^2} F_X \tag{C.4}$$

$$\epsilon = \frac{F_X X}{F} \,. \tag{C.5}$$

C.2 Phonon propagation speeds

To find the phonon propagation speeds, we should expand the solid action (C.2) to quadratic order in the phonon field $\vec{\pi}$. Let's use the same X and b^{ij} variables of last section; the expansion of the action then reads

$$\mathcal{L} = F_X\,\delta X + F_{ij}\,\delta b^{ij} + \tfrac{1}{2}F_{XX}(\delta X)^2 + F_{X,ij}\,\delta X\,\delta b^{ij} + \tfrac{1}{2}F_{ij,kl}\,\delta b^{ij}\delta b^{kl} + \dots \tag{C.6}$$

When we specialize all the derivatives of F to the background, by icosahedral symmetry they must take the form

$$F_{ij}, F_{X,ij} \propto \delta_{ij}\,, \qquad F_{ij,kl} = f_1\,\delta_{ij}\delta_{kl} + f_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{C.7}$$

with generic, time-dependent coefficients. This kills some of the terms in (C.6) because, by definition ((C.1)), the fluctuation of b^{ij} is traceless. We are left with

$$\mathcal{L} \simeq F_X\,\delta X + \tfrac{1}{2}F_{XX}(\delta X)^2 + f_2\,(\delta b^{ij})^2\,. \tag{C.8}$$

We thus need δX up to quadratic order in the phonon field, and δb^{ij} up to linear order. These are

$$\delta X = \pi^{ii}\,, \qquad \delta b^{ij} \simeq \tfrac{1}{3}\big(\pi^{ij} - \tfrac{1}{3}\pi^{kk}\delta^{ij}\big)\,, \tag{C.9}$$

where π^{ij} is the fluctuation of B^{ij},

$$B^{ij} = \delta^{ij} + \pi^{ij}\,, \qquad \pi^{ij} = \partial^i\pi^j + \partial^j\pi^i + \partial_\mu\pi^i\partial^\mu\pi^j\,. \tag{C.10}$$

At quadratic order in $\vec{\pi}$ we get

$$\mathcal{L}_2 = -F_X\big[\,\dot{\vec{\pi}}^2 - c_T^2(\partial_i\pi_j)^2 - (c_L^2 - c_T^2)(\nabla\cdot\vec{\pi})^2\,\big] \tag{C.11}$$

with

$$c_L^2 = 1 + 2\frac{F_{XX}}{F_X} + \frac{8}{27}\frac{f_2}{F_X}, \quad c_T^2 \equiv 1 + \frac{2}{9}\frac{f_2}{F_X}\,. \tag{C.12}$$

This is identical to the $SO(3)$-invariant solid inflation's result, upon identifying

$$f_2\big|_{\text{here}} \quad \leftrightarrow \quad (F_Y + F_Z)\big|_{\text{there}}\,. \tag{C.13}$$

As a consequence, icosahedral inflation still obeys the relation (1.70), and, more in general, is indistinguishable from solid inflation at quadratic level.

Appendix D Exact icosahedral invariant two-point functions of tensor modes

In this Appendix, we provide the icosahedral invariant tensor two-point function without relying on a perturbative analysis. Using the notation in (3.41), let us rewrite $\mathcal{L}_{\rm int}$ in matrix form:

$$
\begin{aligned}
\mathcal{L}_{\rm int} &= \frac{M_{\rm p}^2 \Delta c_\gamma^2}{8} \int \frac{d\tau d^3k}{(2\pi)^3}\, a^2 \begin{pmatrix} \gamma^+(-\vec{k}) & \gamma^-(-\vec{k}) \end{pmatrix} \begin{pmatrix} pp(\vec{k}) & mp(\vec{k}) \\ pm(\vec{k}) & mm(\vec{k}) \end{pmatrix} \begin{pmatrix} \gamma^+(\vec{k}) \\ \gamma^-(\vec{k}) \end{pmatrix} \\
&\equiv \frac{M_{\rm p}^2 \Delta c_\gamma^2}{8} \int \frac{d\tau d^3k}{(2\pi)^3}\, a^2 \gamma^\dagger(\vec{k}) M(\vec{k}) \gamma(\vec{k})\,.
\end{aligned}
\tag{D.1}
$$

where

$$
\gamma(\vec{k}) \equiv \begin{pmatrix} \gamma^+(\vec{k}) \\ \gamma^-(\vec{k}) \end{pmatrix}\,, \quad M(\vec{k}) \equiv \begin{pmatrix} pp(\vec{k}) & mp(\vec{k}) \\ pm(\vec{k}) & mm(\vec{k}) \end{pmatrix}\,. \tag{D.2}
$$

Because of the identities (3.42), we can say that $M(\vec{k})$ is hermitian and thus diagonalizable. We find that this diagonal basis is given by

$$
\gamma_d(\vec{k}) \equiv \begin{pmatrix} \gamma^1(\vec{k}) \\ \gamma^2(\vec{k}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma^-(\vec{k}) - pm(\vec{k})\gamma^+(\vec{k})/|pm(\vec{k})| \\ \gamma^-(\vec{k}) + pm(\vec{k})\gamma^+(\vec{k})/|pm(\vec{k})| \end{pmatrix}\,. \tag{D.3}
$$

In this basis, the matrix products in $\mathcal{L}_{\rm int}$ become diagonal:

$$
\begin{aligned}
\gamma^\dagger(\vec{k}) M(\vec{k}) \gamma(\vec{k}) &= \gamma_d^\dagger(\vec{k}) M'(\vec{k}) \gamma_d(\vec{k}) \qquad \text{(D.4)} \\
&= \gamma_d^\dagger(\vec{k}) \begin{pmatrix} pp(\vec{k}) - |pm(\vec{k})| & 0 \\ 0 & pp(\vec{k}) + |pm(\vec{k})| \end{pmatrix} \gamma_d(\vec{k})\,.
\end{aligned}
$$

The advantage of using this new basis is that we can compute the two-point functions of $\gamma^1(\vec{k})$ and $\gamma^2(\vec{k})$ in the usual way since they are decoupled. After computing these, we can go back to the $\gamma^\pm$ basis and obtain the exact two-point functions of $\gamma^\pm$,

$$\langle \gamma_1^+ \gamma_2^+ \rangle = \frac{H^2}{M_\mathrm{p}^2} \frac{1}{2k_1^3} \times \left(\frac{1}{\left(1 - \frac{\Delta c_\gamma^2}{2}(pp(\vec{k}_1) - |pm(\vec{k}_1)|)\right)^{\frac{3}{2}}} + \frac{1}{\left(1 - \frac{\Delta c_\gamma^2}{2}(pp(\vec{k}_1) + |pm(\vec{k}_1)|)\right)^{\frac{3}{2}}} \right), \tag{D.5}$$

$$\langle \gamma_1^- \gamma_2^+ \rangle = \frac{H^2}{M_\mathrm{p}^2} \frac{pm(\vec{k}_1)}{2k_1^3 |pm(\vec{k}_1)|} \times \left(-\frac{1}{\left(1 - \frac{\Delta c_\gamma^2}{2}(pp(\vec{k}_1) - |pm(\vec{k}_1)|)\right)^{\frac{3}{2}}} + \frac{1}{\left(1 - \frac{\Delta c_\gamma^2}{2}(pp(\vec{k}_1) + |pm(\vec{k}_1)|)\right)^{\frac{3}{2}}} \right), \tag{D.6}$$

and $\langle \gamma_1^- \gamma_2^- \rangle = \langle \gamma_1^+ \gamma_2^+ \rangle$ and $\langle \gamma_1^+ \gamma_2^- \rangle = \langle \gamma_1^- \gamma_2^+ \rangle^*$ [25]. If Δc_γ^2 is small, these exact results reduce to what we found in the perturbative computations. In the small Δc_γ^2 limit, the exact $\langle \gamma_1^+ \gamma_2^+ \rangle$ reduces to

$$\langle \gamma_1^+ \gamma_2^+ \rangle \simeq \frac{H^2}{M_\mathrm{p}^2} \frac{1}{k_1^3} \left(1 + \frac{3\Delta c_\gamma^2}{4} pp(\vec{k}_1) \right) , \tag{D.7}$$

where the second term is equal to (3.33), and the exact $\langle \gamma_1^- \gamma_2^+ \rangle$ reduces to

$$\langle \gamma_1^- \gamma_2^+ \rangle \simeq \frac{H^2}{M_\mathrm{p}^2} \frac{3\Delta c_\gamma^2 \, pm(\vec{k}_1)}{4k_1^3} , \tag{D.8}$$

which is equal to (3.39).

[25] We omitted the standard $(2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2)$ factors.

Appendix E Full expressions for the relevant quantities in Sect. 4

E.1 Tensor modes

$$g^{ij} = \frac{1}{a^2}\left(\delta_{ij} - \gamma_{ij} + \frac{1}{2}\gamma_{im}\gamma_{mj} + \cdots\right), \tag{E.1}$$

$$R = 6(2H^2 + \dot{H}) + \frac{1}{4}\dot{\gamma}_{ij}^2 - \frac{1}{4a^2}(\partial_m\gamma_{ij})^2, \tag{E.2}$$

$$\begin{aligned} R^{ij} = \frac{1}{a^4}\Bigg[& a^2(3H^2+\dot{H})\delta_{ij} + \frac{a^2}{2}\ddot{\gamma}_{ij} + \frac{3a^2H}{2}\dot{\gamma}_{ij} - \frac{1}{2}\nabla^2\gamma_{ij} - a^2(3H^2+\dot{H})\gamma_{ij} \\ & - \frac{1}{4}\partial_i\gamma_{mn}\partial_j\gamma_{mn} - \frac{1}{2}\partial_m\gamma_{in}\partial_m\gamma_{jn} + \frac{a^2(3H^2+\dot{H})}{2}\gamma_{im}\gamma_{mj} \\ & - \frac{3a^2H}{4}\dot{\gamma}_{im}\gamma_{mj} - \frac{3a^2H}{4}\gamma_{im}\dot{\gamma}_{mj} - \frac{a^2}{4}\ddot{\gamma}_{im}\gamma_{mj} - \frac{a^2}{4}\gamma_{im}\ddot{\gamma}_{mj}\Bigg], \end{aligned} \tag{E.3}$$

$$\begin{aligned} R^{ijkl} = \frac{1}{a^6}\Bigg[& a^2H^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \frac{1}{2}(\partial_k\partial_j\gamma_{il} - \partial_k\partial_i\gamma_{jl} - \partial_l\partial_j\gamma_{ik} + \partial_l\partial_i\gamma_{jk}) \\ & + \frac{1}{4}(-\partial_m\gamma_{ik}\partial_m\gamma_{jl} + \partial_m\gamma_{il}\partial_m\gamma_{jk} + 3\partial_k\gamma_{im}\partial_j\gamma_{lm} + 3\partial_i\gamma_{km}\partial_l\gamma_{jm} \\ & - 3\partial_l\gamma_{im}\partial_j\gamma_{km} - 3\partial_i\gamma_{lm}\partial_k\gamma_{jm}) - a^2H^2(\delta_{ik}\gamma_{jl} - \delta_{il}\gamma_{jk} - \delta_{jk}\gamma_{il} + \delta_{jl}\gamma_{ik}) \\ & + \frac{a^2H}{2}(\delta_{ik}\dot{\gamma}_{jl} - \delta_{il}\dot{\gamma}_{jk} - \delta_{jk}\dot{\gamma}_{il} + \delta_{jl}\dot{\gamma}_{ik}) + a^2H^2(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \\ & + \frac{a^2H^2}{2}(\delta_{ik}\gamma_{jm}\gamma_{ml} - \delta_{il}\gamma_{jm}\gamma_{mk} - \delta_{jk}\gamma_{im}\gamma_{ml} + \delta_{jl}\gamma_{im}\gamma_{mk}) \\ & - \frac{a^2H}{4}(\delta_{ik}\dot{\gamma}_{jm}\gamma_{ml} - \delta_{il}\dot{\gamma}_{jm}\gamma_{mk} - \delta_{jk}\dot{\gamma}_{im}\gamma_{ml} + \delta_{jl}\dot{\gamma}_{im}\gamma_{mk}) \\ & - \frac{a^2H}{4}(\delta_{ik}\gamma_{jm}\dot{\gamma}_{ml} - \delta_{il}\gamma_{jm}\dot{\gamma}_{mk} - \delta_{jk}\gamma_{im}\dot{\gamma}_{ml} + \delta_{jl}\gamma_{im}\dot{\gamma}_{mk}) \\ & - \frac{a^2H}{2}(\dot{\gamma}_{ik}\gamma_{jl} - \dot{\gamma}_{il}\gamma_{jk} - \dot{\gamma}_{jk}\gamma_{il} + \dot{\gamma}_{jl}\gamma_{ik}) + \frac{a^2}{4}(\dot{\gamma}_{ik}\dot{\gamma}_{jl} - \dot{\gamma}_{il}\dot{\gamma}_{jk})\Bigg]. \end{aligned} \tag{E.4}$$

E.2 Scalar modes

$$
\begin{aligned}
g^{ij} \to\ & \frac{1}{a^2}\delta^{ij} + \frac{1}{a^2}(\partial^i\pi^j + \partial^j\pi^i) - \frac{1}{a^2}\partial^i B\partial^j B + \frac{1}{a}(\dot{\pi}^i\partial^j B + \dot{\pi}^j\partial^i B) \\
& - \dot{\pi}^i\dot{\pi}^j + \frac{1}{a^2}\partial_k\pi^i\partial_k\pi^j\,, \qquad \text{(E.5)} \\
R \to\ & 6(2H^2 + \dot{H}) - 12(2H^2 + \dot{H})\psi - 6H\dot{\psi} - 6H\nabla^2 B - \frac{2}{a}\nabla^2\dot{B} - \frac{2}{a^2}\nabla^2\psi \\
& + 24(2H^2 + \dot{H})\psi^2 + 24H\psi\dot{\psi} + \frac{12H}{a}\psi\nabla^2 B + \frac{2}{a}\dot{\psi}\nabla^2 B + \frac{4}{a}\psi\nabla^2\dot{B} \\
& + \frac{4}{a^2}\psi\nabla^2\psi - 6(2H^2 + \dot{H})\partial_k B\partial^k B - 6H\partial_k\dot{B}\partial^k\dot{B} + \frac{6H}{a}\partial_k\psi\partial^k B \\
& + \frac{2}{a^2}\partial_k\psi\partial^k\psi + \frac{1}{a^2}\nabla^2 B\nabla^2 B - \frac{1}{a^2}\partial_i\partial_j B\partial^i\partial^j B\,, \qquad \text{(E.6)} \\
R^{ij} \to\ & \frac{3H^2 + \dot{H}}{a^2}\delta^{ij} - \left(\frac{2(3H^2 + \dot{H})}{a^2}\psi + \frac{H}{a^2}\dot{\psi} + \frac{H}{a^3}\nabla^2 B\right)\delta^{ij} - \frac{2H}{a^3}\partial^i\partial^j B \\
& - \frac{1}{a^3}\partial^i\partial^j\dot{B} - \frac{1}{a^4}\partial^i\partial^j\psi + \frac{3H^2 + \dot{H}}{a^2}(\partial^i\pi^j + \partial^j\pi^i) + \left(\frac{4(3H^2 + \dot{H})}{a^2}\psi^2\right. \\
& \left.+\frac{4H}{a^2}\psi\dot{\psi} + \frac{2H}{a^3}\psi\nabla^2 B - \frac{3H^2 + \dot{H}}{a^2}\partial_k B\partial^k B - \frac{H}{a^2}\partial_k B\partial^k\dot{B} + \frac{H}{a^3}\partial_k\psi\partial^k B\right)\delta^{ij} \\
& - \frac{3(H^2 + \dot{H})}{a^2}\partial^i B\partial^j B + \frac{2H}{a^3}\partial^i\psi\partial^j B + \frac{2H}{a^3}\partial^i B\partial^j\psi + \frac{1}{a^4}\partial^i\psi\partial^j\psi \\
& + \frac{4H}{a^3}\psi\partial^i\partial^j B + \frac{1}{a^3}\dot{\psi}\partial^i\partial^j B + \frac{1}{a^4}\nabla^2 B\partial^i\partial^j B - \frac{1}{a^4}\partial_k\partial^i B\partial^k\partial^j B + \frac{2}{a^3}\psi\partial^i\partial^j\dot{B} \\
& + \frac{2}{a^4}\psi\partial^i\partial^j\psi - \left(\frac{2(3H^2 + \dot{H})}{a^2}\psi + \frac{H}{a^2}\dot{\psi} + \frac{H}{a^3}\nabla^2 B\right)(\partial^i\pi^j + \partial^j\pi^i) \\
& - \frac{2H}{a^3}(\partial_k\pi^i\partial^k\partial^j B + \partial_k\pi^j\partial^k\partial^i B) - \frac{1}{a^3}(\partial_k\pi^i\partial^k\partial^j\dot{B} + \partial_k\pi^j\partial^k\partial^i\dot{B}) \\
& - \frac{1}{a^4}(\partial_k\pi^i\partial^k\partial^j\psi + \partial_k\pi^j\partial^k\partial^i\psi) + \frac{3(H^2 + \dot{H})}{a}(\dot{\pi}^i\partial^j B + \dot{\pi}^j\partial^i B) \\
& - \frac{2H}{a^2}(\dot{\pi}^i\partial^j\psi + \dot{\pi}^j\partial^i\psi) - 3(H^2 + \dot{H})\dot{\pi}^i\dot{\pi}^j + \frac{3H^2 + \dot{H}}{a^2}\partial_k\pi^i\partial^k\pi^j\,, \qquad \text{(E.7)}
\end{aligned}
$$

$$
\begin{aligned}
R^{ijkl} \to\ & \frac{H^2}{a^4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \frac{2H^2}{a^4}\psi(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \frac{H^2}{a^4}\Big(\delta_{ik}\partial_j\partial_l B - \delta_{il}\partial_j\partial_k B \\
& - \delta_{jk}\partial_i\partial_l B + \delta_{jl}\partial_i\partial_k B\Big) + \frac{H^2}{a^4}\Big\{\delta_{ik}(\partial_j\pi^l + \partial_l\pi^j) \\
& - \delta_{il}(\partial_j\pi^k + \partial_k\pi^j) - \delta_{jk}(\partial_i\pi^l + \partial_l\pi^i) + \delta_{jl}(\partial_i\pi^k + \partial_k\pi^i)\Big\} \\
& + \frac{4H^2}{a^4}\psi^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \frac{H^2}{a^4}\partial_k B\partial_k B(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
& - \frac{\ddot{a}}{a^5}(\delta_{ik}\partial_j B\partial_l B - \delta_{il}\partial_j B\partial_k B - \delta_{jk}\partial_i B\partial_l B + \delta_{jl}\partial_i B\partial_k B) \\
& + \frac{H}{a^5}\Big\{\delta_{ik}(\partial_j\psi\partial_l B + \partial_l\psi\partial_j B) - \delta_{il}(\partial_j\psi\partial_k B + \partial_k\psi\partial_j B) \\
& - \delta_{jk}(\partial_i\psi\partial_l B + \partial_l\psi\partial_i B) + \delta_{jl}(\partial_i\psi\partial_k B + \partial_k\psi\partial_i B)\Big\} \\
& + \frac{2H}{a^5}(\delta_{ik}\psi\partial_j\partial_l B - \delta_{il}\psi\partial_j\partial_k B - \delta_{jk}\psi\partial_i\partial_l B + \delta_{jl}\psi\partial_i\partial_k B) \\
& + \frac{1}{a^6}(\partial_i\partial_k B\partial_j\partial_l B - \partial_i\partial_l B\partial_j\partial_k B) - \frac{2H^2}{a^4}\psi\Big\{\delta_{ik}(\partial_j\pi^l + \partial_l\pi^j) \\
& - \delta_{il}(\partial_j\pi^k + \partial_k\pi^j) - \delta_{jk}(\partial_i\pi^l + \partial_l\pi^i) + \delta_{jl}(\partial_i\pi^k + \partial_k\pi^i)\Big\} \\
& - \frac{H}{a^5}\Big\{\partial_i\partial_k B(\partial_j\pi^l + \partial_l\pi^j) - \partial_i\partial_l B(\partial_j\pi^k + \partial_k\pi^j) - \partial_j\partial_k B(\partial_i\pi^l + \partial_l\pi^i) \\
& + \partial_j\partial_l B(\partial_i\pi^k + \partial_k\pi^i) + \delta_{ik}(\partial_k\pi^j\partial_k\partial_l B + \partial_k\pi^l\partial_k\partial_j B) \\
& - \delta_{il}(\partial_k\pi^j\partial_k\partial_k B + \partial_k\pi^k\partial_k\partial_j B) - \delta_{jk}(\partial_k\pi^i\partial_k\partial_l B + \partial_k\pi^l\partial_k\partial_i B) \\
& + \delta_{jl}(\partial_k\pi^i\partial_k\partial_k B + \partial_k\pi^k\partial_k\partial_i B)\Big\} + \frac{H^2}{a^4}\Big\{\delta_{ik}\partial_k\pi^j\partial_k\pi^l - \delta_{il}\partial_k\pi^j\partial_k\pi^k \\
& - \delta_{jk}\partial_k\pi^i\partial_k\pi^l + \delta_{jl}\partial_k\pi^i\partial_k\pi^k + \partial_k\pi^i\partial_l\pi^j + \partial_k\pi^i\partial_j\pi^l + \partial_i\pi^k\partial_l\pi^j \\
& \partial_i\pi^k\partial_j\pi^l - \partial_l\pi^i\partial_k\pi^j - \partial_l\pi^i\partial_j\pi^k - \partial_i\pi^l\partial_k\pi^j - \partial_i\pi^l\partial_j\pi^k\Big\} \\
& - \frac{\ddot{a}}{a^3}(\delta_{ik}\dot{\pi}^j\dot{\pi}^l - \delta_{il}\dot{\pi}^j\dot{\pi}^k - \delta_{jk}\dot{\pi}^i\dot{\pi}^l + \delta_{jl}\dot{\pi}^i\dot{\pi}^k) \\
& + \frac{\ddot{a}}{a^4}\Big\{\delta_{ik}(\dot{\pi}^j\partial_l B + \dot{\pi}^l\partial_j B) - \delta_{il}(\dot{\pi}^j\partial_k B + \dot{\pi}^k\partial_j B) - \delta_{jk}(\dot{\pi}^i\partial_l B + \dot{\pi}^l\partial_i B) \\
& + \delta_{jl}(\dot{\pi}^i\partial_k B + \dot{\pi}^k\partial_i B)\Big\} - \frac{H}{a^4}\Big\{\delta_{ik}(\dot{\pi}^j\partial_l\psi + \dot{\pi}^l\partial_j\psi) - \delta_{il}(\dot{\pi}^j\partial_k\psi + \dot{\pi}^k\partial_j\psi) \\
& - \delta_{jk}(\dot{\pi}^i\partial_l\psi + \dot{\pi}^l\partial_i\psi) + -\delta_{jl}(\dot{\pi}^i\partial_k\psi + \dot{\pi}^k\partial_i\psi)\Big\}\,. \qquad \text{(E.8)}
\end{aligned}
$$

Appendix F Curvature perturbation in the presence of the non-minimal interaction

Since we introduced a new interaction to the matter Lagrangian in Sect. 4, the perturbation of a stress-energy tensor changes accordingly. In this Appendix, we would like to provide the correction to the curvature perturbation ζ, due to the new interaction. First, the quadratic matter action is

$$S_m^{(2)} = M_{\mathrm{p}}^2 \int dt\, d^3x\, a^3 \Bigg[-\frac{3H^2}{2}\partial_i B \partial^i B - \dot{H}\,(1+\alpha)\,\partial_i B\partial^i B + \frac{3H^2}{2}\psi^2 + \frac{2\alpha H}{a}\partial_i\psi\partial^i B + 2a\dot{H}\,(1+\alpha)\,\partial_i B\dot{\pi}^i + 2\dot{H}\psi\partial_i\pi^i + 2\alpha H\psi\partial_i\dot{\pi}^i - a^2\dot{H}M_{\mathrm{p}}^2\,(1+\alpha)\left\{\dot{\pi}_i\dot{\pi}^i - \frac{c_T^2}{a^2}(\partial_i\pi_j)^2 - \frac{c_L^2 - c_T^2}{a^2}(\vec{\nabla}\cdot\vec{\pi})^2\right\}\Bigg] . \tag{F.1}$$

The curvature perturbation ζ is defined as

$$\zeta \equiv \frac{A}{2} - H\frac{\delta\rho}{\dot{\bar{\rho}}} \tag{F.2}$$

in a gauge invariant fashion, where $g_{ij} = a(t)^2(\delta_{ij}(1+A)+\partial_i\partial_j\chi)$ and $\delta T_{00} = -\bar{\rho}\delta g_{00} + \delta\rho$. In our gauge, A is zero, so we need only $\delta\rho$. To find it, recall the definition of the stress-energy tensor:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} . \tag{F.3}$$

Finding the general variation with respect to $\delta g_{\mu\nu}$ of the non-minimal term is difficult. Instead, because in our gauge $\delta g_{00} = -2\psi$, we may use the following chain rule trick:

$$\frac{\delta}{\delta g_{00}} = \frac{\delta\psi}{\delta g_{00}}\frac{\delta}{\delta\psi} = -\frac{1}{2}\frac{\delta}{\delta\psi} , \tag{F.4}$$

which gives

$$T^{00} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi} . \tag{F.5}$$

To compute the variation with respect to ψ, we need tadpole terms in ψ as well as the quadratic action, which are:

$$S_m^{(1)} = M_{\mathrm{p}}^3 \int dt\, d^3x\, a^3 \left[-3H^2\psi + 2\dot{H}\partial_i\pi^i - \frac{3\alpha H}{a}\nabla^2 B - \frac{\alpha}{a^2}\nabla^2\psi \right] . \tag{F.6}$$

Taking the variation of (F.1) and (F.6) with respect to ψ gives

$$\frac{\delta(\sqrt{-g}(\mathcal{L}_m^{(2)}+\mathcal{L}_m^{(1)}))}{\delta\psi} = a^3 M_\mathrm{p}^2\left[-3H^2+3H^2\psi+2\dot{H}\partial_i\pi^i-\frac{2\alpha H}{a}\nabla^2 B+2\alpha H\partial_i\dot{\pi}^i\right]. \tag{F.7}$$

Since $\sqrt{-g}=a^3\left(1+\psi-\psi^2/2+\partial_i B\partial^i B/2\right)$, we have

$$\begin{aligned} T^{00} &= -\frac{1}{a^3\left(1+\psi-\frac{\psi^2}{2}+\frac{\partial_i B\partial^i B}{2}\right)}\frac{\delta(\sqrt{-g}(\mathcal{L}_m^{(2)}+\mathcal{L}_m^{(1)}))}{\delta\psi}\,, \\ &\simeq -M_\mathrm{p}^2(1-\psi)\left[-3H^2+3H^2\psi+2\dot{H}\partial_i\pi^i-\frac{2\alpha H}{a}\nabla^2 B+2\alpha H\partial_i\dot{\pi}^i\right], \\ &= M_\mathrm{p}^2\left[3H^2-6H^2\psi-2\dot{H}\partial_i\pi^i+\frac{2\alpha H}{a}\nabla^2 B-2\alpha H\partial_i\dot{\pi}^i\right]. \end{aligned} \tag{F.8}$$

This is the stress-energy tensor with upper indices, so we need to lower the indices:

$$T_{00}=g_{0\mu}g_{0\nu}T^{\mu\nu}=g_{00}^2T^{00}+2g_{00}g_{0i}T^{0i}+g_{0i}g_{0j}T^{ij}\simeq(1+4\psi)T^{00}\,. \tag{F.9}$$

Combining all we have,

$$T_{00}=M_\mathrm{p}^2\left[3H^2+6H^2\psi-2\dot{H}\partial_i\pi^i+\frac{2\alpha H}{a}\nabla^2 B-2\alpha H\partial_i\dot{\pi}^i\right], \tag{F.10}$$

and its perturbation is

$$\delta T_{00}=M_\mathrm{p}^2\left[6H^2\psi-2\dot{H}\partial_i\pi^i+\frac{2\alpha H}{a}\nabla^2 B-2\alpha H\partial_i\dot{\pi}^i\right]. \tag{F.11}$$

With the definition of $\delta\rho$, $\delta T_{00}=-\bar{\rho}\delta g_{00}+\delta\rho$, we thus get the following expression for $\delta\rho$:

$$\delta\rho=M_\mathrm{p}^2\left[-2\dot{H}\partial_i\pi^i+\frac{2\alpha H}{a}\nabla^2 B-2\alpha H\partial_i\dot{\pi}^i\right]. \tag{F.12}$$

Finally, we have

$$\begin{aligned} \zeta &= H\frac{\delta\rho}{\dot{\bar{\rho}}}=-H\frac{\delta\rho}{6M_p^2H\dot{H}}=-\frac{1}{6M_p^2\dot{H}}\delta\rho \\ &= \frac{1}{3}\left[\partial_i\pi^i-\alpha\frac{H}{\dot{H}}\left(\frac{1}{a}\nabla^2 B-\partial_i\dot{\pi}^i\right)\right]. \end{aligned} \tag{F.13}$$

As we already saw, the fields B and ψ are not dynamic in our action. We first need to integrate them out, which gives the solutions (4.43) and (4.44). If we plug those two solutions into (F.13) in Fourier space, we get

$$\begin{aligned}\zeta &= \frac{1}{3}\left[-k\pi_L + \alpha\frac{H}{\dot{H}}\left(\frac{k^2}{a}B - k\dot{\pi}_L\right)\right] \\ &= -\frac{k}{3}\frac{(1-3a^2\dot{H}/k^2)\pi_L + \alpha H\dot{\pi}_L/\dot{H}}{(1+\alpha)-3a^2\dot{H}/k^2} \\ &= -\frac{k}{3}\frac{(1+3\epsilon a^2H^2/k^2)\pi_L + \alpha H\dot{\pi}_L/\dot{H}}{(1+\alpha)+3\epsilon a^2H^2/k^2}\,. \end{aligned} \tag{F.14}$$

Appendix G Time dependence of the coefficients fixed by internal scale invariance

The constraints from exact internal scale invariance, $\phi^I \to \lambda\phi^I$, along with the constraints (4.18) from invariance under time diffeomorphisms, can fix the time dependence of many coefficients in (4.17). In this Appendix, we provide their time dependence.

First we have the following constraints from exact internal scale invariance, (4.24),

$$0 = F_{B1}\,, \tag{G.1}$$

$$0 = 3F_{B2} + F_{B3}\,, \tag{G.2}$$

$$0 = F_{C0} + 2F_{C1} + 6F_{C2}\,, \tag{G.3}$$

$$0 = 2F_{C2} + F_{C3} + 6F_{C4}\,, \tag{G.4}$$

$$0 = 2F_{C1} + 3F_{C3} + 2F_{C5}\,, \tag{G.5}$$

$$0 = F_{R1}\,, \tag{G.6}$$

$$0 = 3F_{R2} + F_{R3}\,, \tag{G.7}$$

$$0 = F_{M0} + 3F_{M1} + \frac{9}{2}F_{M2}\,, \tag{G.8}$$

$$0 = 3F_{M1} + F_{M3} + 2F_{M4} + 3F_{M5}\,, \tag{G.9}$$

$$0 = 3F_{M2} + 2F_{M5} + 6F_{M6}\,. \tag{G.10}$$

By internal scale invariance, some terms (which are proportional to the Hubble constant) in (4.18) become zero. We thus get simplified constraints from time diffeomorphisms invariance:

$$0 = \dot{F}_{B0} + 3\dot{F}_{B1}\,, \tag{G.11a}$$

$$0 = \dot{F}_{C0} + \dot{F}_{C1} + 3\dot{F}_{C2}\,, \tag{G.11b}$$

$$0 = \dot{F}_{R0} + 3\dot{F}_{R1}\,, \tag{G.11c}$$

$$0 = \dot{F}_{M0} + 2\dot{F}_{M1} + 3\dot{F}_{M2}\,, \tag{G.11d}$$

We can utilize these two sets of constraints to derive the time dependence of the coefficients.

G.1 F_B's

From (G.1) and (G.11a), we obtain

$$\dot{F}_{B0} = 0\,, \qquad F_{B0} = \text{const}\,. \tag{G.12}$$

G.2 F_C's

Taking the time derivative of (G.3), we have

$$\begin{aligned} 0 &= \dot{F}_{C0} + 2\dot{F}_{C1} + 6\dot{F}_{C2}\,, \\ &= \dot{F}_{C1} + 3\dot{F}_{C2} \qquad\qquad \because \text{(G.11b)}\,. \end{aligned} \tag{G.13}$$

which implies

$$F_{C1} + 3F_{C2} = \text{const}\,. \tag{G.14}$$

Using this result and (G.11b), we obtain

$$\dot{F}_{C0} = -\dot{F}_{C1} - 3\dot{F}_{C2} = 0\,, \qquad F_{C0} = \text{const}\,. \tag{G.15}$$

Thus we can write $F_{C0} = M_C^2$ where M_C is a constant whose mass dimension is 1. Then, using (G.3), (G.4) and (G.5), we have

$$F_{C1} + 3F_{C2} = -\frac{M_C^2}{2}, \qquad 3F_{C3} + 9F_{C4} + F_{C5} = \frac{M_C^2}{2}\,. \tag{G.16}$$

G.3 F_R's

From (G.6) and (G.11c), we obtain

$$\dot{F}_{R0} = 0\,, \qquad F_{R0} = \text{const}\,. \tag{G.17}$$

G.4 F_M's

Taking the time derivative of (G.8), we have

$$\begin{aligned} 0 &= \dot{F}_{M0} + 3\dot{F}_{M1} + \frac{9}{2}\dot{F}_{M2}\,, \\ &= \dot{F}_{M1} + \frac{3}{2}\dot{F}_{M2} \qquad\qquad \because \text{(G.11d)}\,. \end{aligned} \tag{G.18}$$

We thus have

$$\dot{F}_{M1} + \frac{3}{2}\dot{F}_{M2} = \text{const}\,. \tag{G.19}$$

Using this result and (G.11d), we obtain

$$\dot{F}_{M0} = -2\dot{F}_{M1} - 3\dot{F}_{M2} = 0\,, \qquad F_{M0} = \text{const}\,. \tag{G.20}$$

Hence, we can write $F_{M0} = M_M^2$, where M_M is a constant whose mass dimension is 1. Then, using (G.8) and (G.9), we have

$$F_{M1} + \frac{3}{2}F_{M2} = -\frac{M_M^2}{3}, \qquad F_{M3} + 2F_{M4} + 6F_{M5} + 9F_{M6} = M_M^2\,. \tag{G.21}$$

G.5 Summary

In the exact de Sitter limit,

$$\begin{aligned}
&F_{B0}(t) = F_{B0}, \quad && F_{B1}(t) = 0, && 3F_{B2}(t) + F_{B3}(t) = 0, \\
&F_{C0}(t) = M_C^2, && F_{C1}(t) + 3F_{C2}(t) = -\tfrac{M_C^2}{2}, && 3F_{C3}(t) + 9F_{C4}(t) + F_{C5}(t) = \tfrac{M_C^2}{2}, \\
&F_{R0}(t) = M_R^2, && F_{R1}(t) = 0, && 3F_{R2}(t) + F_{R3}(t) = 0, \\
&F_{M0}(t) = M_M^2, && F_{M1}(t) + \tfrac{3}{2}F_{M2}(t) = -\tfrac{M_M^2}{3}, && F_{M3}(t) + 2F_{M4}(t) \\
& && && +6F_{M5}(t) + 9F_{M6}(t) = M_M^2.
\end{aligned} \tag{G.22}$$

Appendix H Non-minimal coupling terms with Ricci scalar and Riemann tensor

In this Appendix, we provide the expressions for $\mathcal{L}_R$ and $\mathcal{L}_M$ in (4.17), which we had omitted for simplicity.

H.1 Scalar modes

Without imposing any constraints, $\mathcal{L}_R$ and $\mathcal{L}_M$ are

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_R = a^3 \Bigg[& \frac{4H}{a} F_{R0} \partial_i \psi \partial^i B + (F_{R0} - 3F_{R1}) \left(-9H^2\psi^2 + 3H^2 \partial_i B \partial^i B\right) \\
& - 4F_{R1} \partial_i \pi^i \left(3H^2 \psi + \frac{3H}{a} \nabla^2 B + \frac{1}{a^2} \nabla^2 \psi \right) \\
& + 4F_{R1} \partial_i \dot{\pi}^i \left(3H\psi + \frac{1}{a} \nabla^2 B \right) + 4\dot{F}_{R1} \partial_i \pi^i \left(3H\psi + \frac{1}{a} \nabla^2 B \right) \\
& + 4H F_{R1} \partial_i \pi^i \left(3H\psi + \frac{1}{a} \nabla^2 B \right) + 12a(2H^2 + \dot{H}) F_{R1} \dot{\pi}^i \partial_i B \\
& - 6(2H^2 + \dot{H}) F_{R1} \partial_i B \partial^i B - 6(2H^2 + \dot{H}) \Big\{ F_{R1} \dot{\vec{\pi}}^2 \\
& - (F_{R1} + 2F_{R3})(\partial_i \pi_j)^2 - (4F_{R2} + 2F_{R3})(\partial_i \pi^i)^2 \Big\} \Bigg] ,
\end{aligned}
\tag{H.1}
$$

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_M =a^3\Bigg[&18H^2\left(F_{M0}+2F_{M1}+3F_{M2}\right)\psi^2-2(7H^2+4\dot{H})F_{M0}\partial_iB\partial^iB\\
&-4(9H^2+4\dot{H})F_{M1}\partial_iB\partial^iB-6(9H^2+4\dot{H})F_{M2}\partial_iB\partial^iB\\
&-16H^2\left(F_{M0}+3F_{M1}+\frac{9}{2}F_{M2}\right)\psi\partial_i\pi^i\\
&-\frac{16H}{a}\left(F_{M0}+3F_{M1}+4F_{M2}\right)\nabla^2B\partial_i\pi^i\\
&+\frac{8H}{a}\left(F_{M0}+2F_{M1}+3F_{M2}\right)\partial_i\psi\partial^iB+16a(H^2+\dot{H})F_{M0}\dot{\pi}^i\partial_iB\\
&+16a(3H^2+2\dot{H})F_{M1}\dot{\pi}^i\partial_iB+24a(3H^2+2\dot{H})F_{M2}\dot{\pi}^i\partial_iB\\
&-16H\left(F_{M0}+2F_{M1}+3F_{M2}\right)\dot{\pi}^i\partial_i\psi\\
&-a^2\left(8(H^2+\dot{H})F_{M0}+8(3H^2+2\dot{H})F_{M1}+12(3H^2+2\dot{H})F_{M2}\right)\dot{\vec{\pi}}^2\\
&+4H^2\bigg\{\left(F_{M0}+6F_{M1}+6F_{M2}-F_{M3}+4F_{M4}\right)(\partial_i\pi_j)^2\\
&+\left(F_{M0}+8F_{M1}+11F_{M2}+F_{M3}+4F_{M4}+8F_{M5}+12F_{M6}\right)(\partial_i\pi^i)^2\bigg\}\Bigg].
\end{aligned}
\tag{H.2}
$$

As we did in Sect. 4.4.1, let us impose exact scale invariance on $\mathcal{L}_R$ and $\mathcal{L}_M$. Then, we have

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_R =a^3\Bigg[&\frac{4H}{a}F_{R0}\partial_i\psi\partial^iB+F_{R0}\left(-9H^2\psi^2+3H^2\partial_iB\partial_iB\right)\\
&-12(2H^2+\dot{H})F_{R2}\left\{3(\partial_i\pi_j)^2+(\partial_i\pi^i)^2\right\}\Bigg],
\end{aligned}
\tag{H.3}
$$

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_M =a^3\Bigg[&6H^2F_{M0}\psi^2-2H^2F_{M0}\partial_iB\partial^iB-\frac{8\dot{H}}{3}F_{M0}\partial_iB\partial^iB+\frac{8H}{a}F_{M2}\nabla^2B\partial_i\pi^i\\
&+\frac{8H}{3a}F_{M0}\partial_i\psi\partial^iB+\frac{16a\dot{H}}{3}F_{M0}\dot{\pi}^i\partial_iB-\frac{16H}{3}F_{M0}\dot{\pi}^i\partial_i\psi-\frac{8a^2\dot{H}}{3}F_{M0}\dot{\vec{\pi}}^2\\
&-4H^2\left(\frac{2}{3}F_{M3}-\frac{2}{3}F_{M4}+\frac{4}{3}F_{M5}+F_{M6}\right)\left\{3(\partial_i\pi_j)^2+(\partial_i\pi^i)^2\right\}\Bigg].
\end{aligned}
\tag{H.4}
$$

Adding all of these and $\mathcal{L}_{\rm EH}$, $\mathcal{L}_B$ and $\mathcal{L}_C$ by setting $F_{C0}\equiv M_C^2$, $F_{R0}\equiv M_R^2$ and

$F_{M0} \equiv M_M^2,$

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_{\rm tot} = a^3 \Bigg[& -\dot{H}\left(M_{\rm p}^2 + M_4^2 + 2M_5^2 + \frac{4M_6^2}{3}\right)\partial_i B \partial^i B \\
& - 3H^2\left(M_{\rm p}^2 + 2M_5^2 - \frac{4M_6^2}{3}\right)\psi^2 \\
& + \frac{2H}{a}\left(M_{\rm p}^2 + M_4^2 + 2M_5^2 + \frac{4M_6^2}{3}\right)\partial_i\psi\partial^i B \\
& + 2a\dot{H}\left(M_{\rm p}^2 + M_4^2 + 2M_5^2 + \frac{4M_6^2}{3}\right)\partial_i B \dot{\pi}^i \\
& + 2\dot{H}\left(M_{\rm p}^2 + 2M_5^2 - \frac{4M_6^2}{3}\right)\psi\partial_i\pi^i + 2H\left(M_4^2 + \frac{8M_6^2}{3}\right)\psi\partial_i\dot{\pi}^i \\
& + \left(-\frac{4}{a}\dot{F}_{42} + \frac{8H}{a}F_{M2}\right)\nabla^2 B\partial_i\pi^i - \frac{4F_{C2}}{a}\nabla^2 B\partial_i\dot{\pi}^i + \frac{4F_{C2}}{a^2}\nabla^2\psi\partial_i\pi^i \\
& - a^2\dot{H}\left(M_{\rm p}^2 + M_4^2 + 2M_5^2 + \frac{4M_6^2}{3}\right)\dot{\vec{\pi}}^2 \\
& + \Bigg\{\frac{1}{3}F_X X + \frac{2}{9}(F_Y + F_Z) - 3\left(3H^2 + \dot{H}\right)F_{C3} - 36\left(2H^2 + \dot{H}\right)F_{R2} \\
& -12H^2\left(\frac{2}{3}F_{M3} - \frac{2}{3}F_{M4} + \frac{4}{3}F_{M5} + F_{M6}\right)\Bigg\}\left(\partial_i\pi_j\right)^2 \\
& + \Bigg\{\frac{2}{9}F_{XX}X^2 + \frac{2}{27}(F_Y + F_Z) - \left(3H^2 + \dot{H}\right)F_{C3} - 12\left(2H^2 + \dot{H}\right)F_{R2} \\
& -4H^2\left(\frac{2}{3}F_{M3} - \frac{2}{3}F_{M4} + \frac{4}{3}F_{M5} + F_{M6}\right)\Bigg\}\left(\partial_i\pi^i\right)^2\Bigg] \,. \qquad \text{(H.5)}
\end{aligned}
$$

Notice that for the minimal coupling Lagrangian $\mathcal{L}_B$, we use X, Y and Z as the building blocks instead of using the form we had before, (4.17).

H.2 Tensor modes

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_R = \; & a^3 \left[\frac{1}{4} \left(F_{R0} + 3F_{R1}\right) \left(\dot{\gamma}_{ij}\right)^2 - \frac{1}{4a^2} \left(F_{R0} + 3F_{R1}\right) \left(\partial_m \gamma_{ij}\right)^2 \right. \\
& \left. + 6(2H^2 + \dot{H}) \left(\frac{F_{R1}}{2} + F_{R3} \right) \left(\gamma_{ij}\right)^2 \right] , \qquad \text{(H.6)} \\
\sqrt{-g}\mathcal{L}_M = \; & a^3 \left[-\frac{1}{2} \left(F_{M0} + 2F_{M1} + 3F_{M2}\right) \left(\dot{\gamma}_{ij}\right)^2 \right. \\
& - \frac{1}{2a^2} \left(5F_{M0} + 14F_{M1} + 15F_{M2}\right) \left(\partial_m \gamma_{ij}\right)^2 \\
& + 2H^2 \left(F_{M0} + 6F_{M1} 6F_{M2} - F_{M3} + 4F_{M4}\right) \left(\gamma_{ij}\right)^2 \\
& + (3H^2 + \dot{H}) \left(F_{M0} + 3F_{M1} + 3F_{M2}\right) \left(\gamma_{ij}\right)^2 \\
& \left. + H \frac{\partial}{\partial t} \left(F_{M0} + 3F_{M1} + 3F_{M2}\right) \left(\gamma_{ij}\right)^2 \right] . \qquad \text{(H.7)}
\end{aligned}
$$

Appendix I Corrections to power spectra due to the small non-minimal interaction

From (G.22) and the arguments made in Sect. 4.4.1 to get (4.39), the exact, scale-invariant, non-minimal interaction that makes sense physically is

$$
\begin{aligned}
\mathcal{L}_C &= M_C^2 a^2 \left(\delta^{IJ} - \frac{a^2}{2} g^{IJ} + \frac{a^4}{2} \delta g^{IK} \delta g^{KJ} + \cdots \right) R^{IJ} , && \text{(I.1)} \\
&= \frac{M_C^2}{2} a^2 \left(\delta^{IJ} - a^2 \delta g^{IJ} + a^4 \delta g^{IK} \delta g^{KJ} + \cdots \right) R^{IJ} , && \text{(I.2)} \\
&= \frac{M_C^2}{2} \left(g^{-1}\right)^{IJ} R^{IJ} && \text{(I.3)}
\end{aligned}
$$

in unitary gauge. If we restore the phonon fields, π^I, via the Stückelberg mechanism, this non-minimal interaction can be written as

$$
\mathcal{L}_C = \frac{M_C^2}{2} (B^{-1})^{IJ} C^{IJ} = \frac{M_C^2}{2} (B^{-1})^{IJ} R^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J , \qquad \text{(I.4)}
$$

where $\phi^I = x^I + \pi^I$. Then, the full action with the Einstein-Hilbert and the minimal interaction is

$$\begin{aligned} S &= S_{\rm EH} + \int d^4x \sqrt{-g}\left[F(X,Y,Z) + \frac{M_C^2}{2} R^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J(B^{-1})^{IJ}\right] , \\ &\equiv S_{\rm EH} + S_m + \frac{M_C^2}{2}\int d^4x\sqrt{-g}\, R^{\mu\nu}C_{\mu\nu} , \end{aligned} \tag{I.5}$$

where $C_{\mu\nu} \equiv \partial_\mu\phi^I\partial_\nu\phi^J(B^{-1})^{IJ}$.

In this Appendix, we assume M_C^2 is very small and treat the non-minimal interaction as a small perturbation around the minimal interaction, $F(X,Y,Z)$. Then, we compute the corrections to the scalar and tensor power spectra and check whether these results are consistent with what we had in Sect. 4.4.1 and 4.4.2 in the small M_C^2 limit.

The idea is the following: Without $\mathcal{L}_C$, the equation of motion for the metric is the usual Einstein equation. Because $\mathcal{L}_C$ is linear in the Ricci tensor, $R^{\mu\nu}$, we might somehow be able to rewrite $\mathcal{L}_C$ in terms of the Einstein equation with only $F(X,Y,Z)$ and some non-trivial function of $F(X,Y,Z)$ from the stress-energy tensor in the small M_C^2 limit. Then, upon redefinition of the metric, $\mathcal{L}_C$ can be absorbed into the minimal interaction, $F(X,Y,Z)$.

First, let us work on the Einstein equation with $F(X,Y,Z)$. By varying $S_{\rm EH}$ with respect to $g_{\mu\nu}$, we have

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}} , \tag{I.6}$$

and its trace is

$$R - 2R = -R = -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}}g_{\mu\nu} . \tag{I.7}$$

Thus (I.6) can be written as

$$
\begin{aligned}
R^{\mu\nu} &= \frac{1}{2}g^{\mu\nu}R - \frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}} \\
&= -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\left[-\frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\frac{\delta S_{\rm EH}}{\delta g_{\alpha\beta}} + \frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}}\right] \\
&= -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g_{\alpha\beta}}\left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right) \\
&\equiv -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g_{\alpha\beta}}D^{\mu\nu}_{\alpha\beta} \qquad\qquad \text{where } D^{\mu\nu}_{\alpha\beta} \equiv \delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta} \\
&= -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\left(\frac{\delta S_0}{\delta g_{\alpha\beta}} - \frac{\delta S_m}{\delta g_{\alpha\beta}}\right)D^{\mu\nu}_{\alpha\beta} \qquad \text{where } S_0 \equiv S_{\rm EH} + S_m \\
&= -\frac{1}{M_{\rm p}^2}\frac{2}{\sqrt{-g}}\frac{\delta S_0}{\delta g_{\alpha\beta}}D^{\mu\nu}_{\alpha\beta} + \frac{1}{M_{\rm p}^2}T^{\alpha\beta}D^{\mu\nu}_{\alpha\beta}\,.
\end{aligned}
\tag{I.8}
$$

Using this, we can rewrite the original action (I.5) as follows:

$$
S = S_0 - \int d^4x \frac{M_C^2}{M_{\rm p}^2}\frac{\delta S_0}{\delta g_{\alpha\beta}}D^{\mu\nu}_{\alpha\beta}C_{\mu\nu} + \frac{M_4^2}{2M_{\rm p}^2}\int d^4x\sqrt{-g}\,T^{\alpha\beta}D^{\mu\nu}_{\alpha\beta}C_{\mu\nu}\,. \tag{I.9}
$$

Here, at the linear order in M_C^2, the second term can be removed by making the following metric redefinition:

$$
\hat{g}_{\alpha\beta} \equiv g_{\alpha\beta} - \frac{M_C^2}{M_{\rm p}^2}D^{\mu\nu}_{\alpha\beta}C_{\mu\nu} \equiv g_{\alpha\beta} - \alpha D^{\mu\nu}_{\alpha\beta}C_{\mu\nu}\,, \tag{I.10}
$$

and we have

$$
S = S_0(\hat{g}) + \frac{\alpha}{2}\int d^4x\sqrt{-\hat{g}}\,T^{\alpha\beta}D^{\mu\nu}_{\alpha\beta}C_{\mu\nu}(\hat{g})\,. \tag{I.11}
$$

I.1 Detailed computation

Let us take a closer look at the second term of (I.10),

$$
\begin{aligned}
D^{\mu\nu}_{\alpha\beta}C_{\mu\nu} &= \left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right)\partial_\mu\phi^I\partial_\nu\phi^J(B^{-1})^{IJ} \\
&= \partial_\alpha\phi^I\partial_\beta\phi^J(B^{-1})^{IJ} - \frac{1}{2}g_{\alpha\beta}B^{IJ}(B^{-1})^{IJ} \\
&= \partial_\alpha\phi^I\partial_\beta\phi^J(B^{-1})^{IJ} - \frac{3}{2}g_{\alpha\beta}\,.
\end{aligned}
\tag{I.12}
$$

Therefore, the new metric $\hat{g}_{\alpha\beta}$ is

$$\hat{g}_{\alpha\beta} = \left(1 + \frac{3\alpha}{2}\right) g_{\alpha\beta} - \alpha \partial_\alpha \phi^I \partial_\beta \phi^J (B^{-1})^{IJ}\,, \tag{I.13}$$

and

$$T^{\alpha\beta} D^{\mu\nu}_{\alpha\beta} C_{\mu\nu} = T^{\alpha\beta} \partial_\alpha \phi^I \partial_\beta \phi^J (B^{-1})^{IJ} - \frac{3}{2} T\,. \tag{I.14}$$

In [52], the energy-momentum tensor in terms of F is

$$T_{\mu\nu} = g_{\mu\nu} F - 2\partial_\mu \phi^I \partial_\nu \phi^J \left(\left(F_X - \frac{2F_Y Y}{X} - \frac{3F_Z Z}{X} \right) \delta^{IJ} + \frac{2F_Y B^{IJ}}{X^2} + \frac{3F_Z B^{IK} B^{KJ}}{X^3} \right) \tag{I.15}$$

and its trace is

$$\begin{aligned} T &= 4F - 2B^{IJ} \left(\left(F_X - \frac{2F_Y Y}{X} - \frac{3F_Z Z}{X} \right) \delta^{IJ} + \frac{2F_Y B^{IJ}}{X^2} + \frac{3F_Z B^{IK} B^{KJ}}{X^3} \right) \\ &= 4F - 2\left(\left(F_X - \frac{2F_Y Y}{X} - \frac{3F_Z Z}{X} \right) X + 2F_Y Y + 3F_Z Z \right) \\ &= 4F - 2F_X X\,. \end{aligned} \tag{I.16}$$

The first term of (I.14) is

$$\begin{aligned} & T^{\alpha\beta} \partial_\alpha \phi^I \partial_\beta \phi^J (B^{-1})^{IJ} \\ & = F B^{IJ} (B^{-1})^{IJ} - 2 g^{\alpha\mu} g^{\beta\nu} \partial_\mu \phi^L \partial_\nu \phi^M \partial_\alpha \phi^I \partial_\beta \phi^J (B^{-1})^{IJ} \\ & \quad \times \left(\left(F_X - \frac{2F_Y Y}{X} - \frac{3F_Z Z}{X} \right) \delta^{LM} + \frac{2F_Y B^{LM}}{X^2} + \frac{3F_Z B^{LK} B^{KM}}{X^3} \right) \\ & = 3F - 2(B^{-1})^{IJ} B^{IL} B^{JM} \cdot \left(\cdots \right) \\ & = 3F - 2B^{LM} \left(\left(F_X - \frac{2F_Y Y}{X} - \frac{3F_Z Z}{X} \right) \delta^{LM} + \frac{2F_Y B^{LM}}{X^2} + \frac{3F_Z B^{LK} B^{KM}}{X^3} \right) \\ & = 3F - 2F_X X\,. \end{aligned} \tag{I.17}$$

Therefore, (I.14) is

$$T^{\alpha\beta} D^{\mu\nu}_{\alpha\beta} C_{\mu\nu} = 3F - 2F_X X - \frac{3}{2}(4F - 2F_X X) = -3F + F_X X\,. \tag{I.18}$$

Collecting everything,

$$S = S_0(\hat{g}) + \frac{\alpha}{2}\int d^4x\sqrt{-\hat{g}}\Big[-3F + F_X X\Big](\hat{g})\,, \tag{I.19}$$

where

$$\hat{g}_{\alpha\beta} = \left(1+\frac{3\alpha}{2}\right)g_{\alpha\beta} - \alpha\partial_\alpha\phi^I\partial_\beta\phi^J(B^{-1})^{IJ}\,. \tag{I.20}$$

Therefore, in this new language, we can think of the solid Lagrangian

$$F_{\rm new}(\hat{g}) \equiv \left(1-\frac{3\alpha}{2}\right)F(\hat{g}) + \frac{\alpha}{2}F_X(\hat{g})X(\hat{g})\,. \tag{I.21}$$

I.2 New coordinate x'

Notice that $\hat{g}$ does not have the form of the FRW metric on the background,

$$\hat{g}_{\mu\nu}\,dx^\mu dx^\nu = -\left(1+\frac{3\alpha}{2}\right)dt^2 + \left(1+\frac{\alpha}{2}\right)a^2(t)dx^2\,. \tag{I.22}$$

Therefore, for consistency, we have to define the new coordinates in the following way to get the FRW background:

$$t' \equiv \left(1+\frac{3\alpha}{4}\right)t\,, \qquad x' \equiv \left(1+\frac{\alpha}{4}\right)x\,. \tag{I.23}$$

Then in this coordinate system, the line element is

$$\hat{g'}_{\mu\nu}(x')dx'^\mu dx'^\nu = -dt'^2 + a'^2(t')dx'^2\,, \tag{I.24}$$

where $a'(t') \equiv a(t)$. More explicitly,

$$\hat{g'}_{00}(x') = \left(1-\frac{3\alpha}{2}\right)\hat{g}_{00}\,, \tag{I.25}$$

$$\hat{g'}_{0i}(x') = (1-\alpha)\,\hat{g}_{0i}\,, \tag{I.26}$$

$$\hat{g'}_{ij}(x') = \left(1-\frac{\alpha}{2}\right)\hat{g}_{ij}\,. \tag{I.27}$$

Since this transformation is just a particular type of diffeomorphism, the action, (I.19), should be invariant:

$$S = S_0(\hat{g}') + \frac{\alpha}{2}\int d^4x'\sqrt{-\hat{g}'}\Big[-3F + F_X X\Big](\hat{g}')\,. \tag{I.28}$$

From now on, we will use this new metric with the new coordinates.

I.3 New background: the Hubble constant

I.3.1 Method 1

The new Hubble constant is defined as

$$H'(t') = \frac{da'(t')/dt'}{a'(t')} = \left(1 - \frac{3\alpha}{4}\right)\frac{da(t)/dt}{a(t)} = \left(1 - \frac{3\alpha}{4}\right)H(t)\,. \tag{I.29}$$

The time derivative of the Hubble constant can be derived in a similar way,

$$\begin{aligned}\dot{H}'(t') = dH'(t')/dt' &= \left(1 - \frac{3\alpha}{4}\right)dH'(t')/dt \\ &= \left(1 - \frac{3\alpha}{2}\right)dH(t)/dt = \left(1 - \frac{3\alpha}{2}\right)\dot{H}(t)\,.\end{aligned} \tag{I.30}$$

I.3.2 Method 2

In solid inflation, the Hubble constant squared is proportional to the solid Lagrangian F on the background where $X = 3/a^2(t)$. With the new metric and coordinates, we have the following solid Lagrangian:

$$\int d^4x'\sqrt{-\hat{g}'}\left[\left(1 - \frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X\right]\,. \tag{I.31}$$

Therefore, the square of the Hubble constant is

$$H'^2(t') \sim \left(1 - \frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X\,, \tag{I.32}$$

where everything is evaluated on the background. However, with the new metric, the value of X is not $3/a'^2(t')$ but $3/a'^2(t')\left(1 - \frac{\alpha}{2}\right)$:

$$X = \left(1 - \frac{\alpha}{2}\right)X_0 = X_0 - \frac{\alpha}{2}X_0\,, \tag{I.33}$$

where $X_0 = 3/a'^2(t') = 3/a^2(t)$. As a result, we have

$$H'^2(t') \sim \left(1 - \frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X = \left(1 - \frac{3\alpha}{2}\right)F(X_0) \sim \left(1 - \frac{3\alpha}{2}\right)H^2\,, \tag{I.34}$$

which is consistent with Method 1. A similar proof can be made for $\dot{H}$.

I.4 New metric perturbation

Our gauge is

$$g_{00} = -1 - 2\psi\,, \tag{I.35}$$

$$g_{0i} = a\partial_i B\,, \tag{I.36}$$

$$g_{ij} = a^2 \exp\left(\gamma_{ij}\right). \tag{I.37}$$

The new metric as a function of the old coordinates, $\hat{g}(x)$, at the linear order in fluctuations is

$$\begin{aligned}
\hat{g}_{00}(x) &= \left(1 + \frac{3\alpha}{2}\right) g_{00}(x) - \alpha\dot{\pi}^I\dot{\pi}^J\left(B^{-1}\right)^{IJ} \\
&= -\left(1 + \frac{3\alpha}{2}\right)\left(1 + 2\psi(x)\right), \qquad \text{(I.38)}\\
\hat{g}_{0i}(x) &= \left(1 + \frac{3\alpha}{2}\right) g_{0i}(x) - \alpha\dot{\pi}^I\partial_i\phi^J\left(B^{-1}\right)^{IJ} \\
&= \left(1 + \frac{3\alpha}{2}\right) a\partial_i B - \alpha a^2\dot{\pi}_i\,, \qquad \text{(I.39)}\\
\hat{g}_{ij}(x) &= \left(1 + \frac{3\alpha}{2}\right) g_{ij}(x) - \alpha\left(\delta_i^I + \partial_i\pi^I\right)\left(\delta_j^J + \partial_j\pi^J\right) \\
&\qquad\qquad \times a^2\left(\delta^{IJ} + \gamma_{IJ} - \partial_I\pi_J - \partial_J\pi_I\right) \\
&= \left(1 + \frac{3\alpha}{2}\right) a^2\left(\delta_{ij} + \gamma_{ij}\right) - \alpha a^2\left(\delta_{ij} + \gamma_{ij}\right) \\
&= \left(1 + \frac{\alpha}{2}\right) a^2\left(\delta_{ij} + \gamma_{ij}\right). \qquad \text{(I.40)}
\end{aligned}$$

Using these relations, the new metric as a function of the new coordinates $\hat{g}'(x')$ at the linear order in fluctuations can be written as

$$\hat{g}'_{ij}(x') = \left(1 - \frac{\alpha}{2}\right)\hat{g}_{ij}(x) = a^2\left(\delta_{ij} + \gamma_{ij}(x)\right)\,, \tag{I.41}$$

$$\hat{g}'_{0i}(x') = (1-\alpha)\,\hat{g}_{0i}(x) = \left(1 + \frac{\alpha}{2}\right) a\partial_i B - \alpha a^2\dot{\pi}_i\,, \tag{I.42}$$

$$\hat{g}'_{00}(x') = \left(1 - \frac{3\alpha}{2}\right)\hat{g}_{00}(x) = -1 - 2\psi(x)\,. \tag{I.43}$$

We thus have

$$\gamma'_{ij}(x') = \gamma_{ij}(x)\,, \tag{I.44}$$

$$\partial'_i B'(x') = \left(1+\frac{\alpha}{2}\right)\partial_i B - \alpha a \dot{\pi}_i\,, \tag{I.45}$$

$$\psi'(x') = \psi(x)\,. \tag{I.46}$$

I.5 Phonon speeds

B^{IJ} with the new metric and coordinates is [26]:

$$\hat{B'}^{IJ} = \frac{1-\frac{\alpha}{2}}{a'^2}\Big[\delta^{IJ} + \left(1+\frac{\alpha}{4}\right)\left(\partial^I\pi^J + \partial^J\pi^I\right) - \left(1+\frac{\alpha}{2}\right)\left(a'^2\dot{\pi}^I\dot{\pi}^J - \partial_k\pi^I\partial_k\pi^J\right)\Big]\,. \tag{I.47}$$

For simplicity, let me drop the hat and prime from now on. Given this B^{IJ}, we have the following:

$$X = \frac{3}{a^2}\left(1-\frac{\alpha}{2}\right)\left[1+\frac{2}{3}\left(1+\frac{\alpha}{4}\right)\partial_i\pi^i - \frac{1}{3}\left(1+\frac{\alpha}{2}\right)\left(a^2\dot{\vec{\pi}}^2 - (\partial_i\pi_j)^2\right)\right]\,, \tag{I.48}$$

$$Y = \frac{1}{3}\left[1+\frac{2}{9}\left(1+\frac{\alpha}{2}\right)(\partial_i\pi^i)^2 + \frac{2}{3}\left(1+\frac{\alpha}{2}\right)(\partial_i\pi_j)^2\right]\,, \tag{I.49}$$

$$Z = \frac{1}{9}\left[1+\frac{2}{3}\left(1+\frac{\alpha}{2}\right)(\partial_i\pi^i)^2 + 2\left(1+\frac{\alpha}{2}\right)(\partial_i\pi_j)^2\right]\,. \tag{I.50}$$

Using these building blocks, the first term of (I.31) gives us

$$\begin{aligned}\mathcal{L}_{2,(1)} \sim (1-\alpha)a^2\Big[&-\frac{1}{3}F_X X\dot{\vec{\pi}}^2 + \frac{1}{a^2}\left(\frac{1}{3}F_X X + \frac{2}{9}(F_Y+F_Z)\right)(\partial_i\pi_j)^2 \\ &+\frac{1}{a^2}\left(\frac{2}{9}F_{XX}X^2 + \frac{2}{27}(F_Y+F_Z)\right)(\partial_i\pi^i)^2\Big]\,,\end{aligned} \tag{I.51}$$

and the second term gives us

$$\begin{aligned}\mathcal{L}_{2,(2)} \sim \frac{\alpha}{2}a^2\Big[&-\frac{1}{3}F_{XX}X^2\dot{\vec{\pi}}^2 + \frac{1}{a^2}\left(\frac{1}{3}F_{XX}X^2 + \frac{2}{9}(F_{XY}+F_{XZ})X\right)(\partial_i\pi_j)^2 \\ &+\left(\frac{2}{9}F_{XXX}X^3 + \frac{2}{27}(F_{XY}+F_{XZ})X\right)(\partial_i\pi^i)^2\Big] \\ +\frac{\alpha}{2}a^2\Big[&-\frac{1}{3}F_X X\dot{\pi}^I\dot{\pi}^I + \frac{1}{a^2}\left(\frac{1}{3}F_X X\right)(\partial_i\pi_j)^2 + \frac{1}{a^2}\frac{4}{9}F_{XX}X^2(\partial_i\pi^i)^2\Big]\,.\end{aligned} \tag{I.52}$$

[26]Below, all derivatives are with respect to x'. We omit the prime for convenience. For instance, $\partial_i\pi^i = \partial'_i\pi'^i$.

Again using $X = X_0 - \frac{\alpha}{2} X_0$, we have

$$\begin{aligned}
\frac{1}{3} F_X X &= \frac{1}{3}\left(F_X(X_0) - \frac{\alpha}{2} F_{XX}(X_0) X_0\right)\left(X_0 - \frac{\alpha}{2} X_0\right) \\
&= \frac{1}{3}\left(F_X(X_0) X_0 - \frac{\alpha}{2} F_X(X_0) X_0 - \frac{\alpha}{2} F_{XX}(X_0) X_0^2\right)\,, \qquad \text{(I.53)} \\
\frac{2}{9} F_{XX} X^2 &= \frac{2}{9}\left(F_X X(X_0) - \frac{\alpha}{2} F_{XXX}(X_0) X_0\right)\left(X_0^2 - \alpha X_0^2\right) \\
&= \frac{2}{9}\left(F_{XX}(X_0) X_0^2 - \frac{\alpha}{2} F_{XXX}(X_0) X_0^3 - \alpha F_{XX}(X_0) X_0^2\right)\,, \qquad \text{(I.54)} \\
F_Y + F_Z &= \left(F_Y + F_Z\right)(X_0) - \frac{\alpha}{2}\left(F_{XZ} + F_{YZ}\right)(X_0) X_0\,, \qquad \text{(I.55)}
\end{aligned}$$

and the zeroth order term of (I.51) gives

$$\begin{aligned}
\mathcal{L}_{2,(1)}\left[\mathcal{O}(0)\right] = \mathcal{L}_{2,(1)}\left[\mathcal{O}(0)\right](X_0) + \frac{\alpha}{2}\Bigg[&\left(\frac{1}{3} F_X X + \frac{1}{3} F_{XX} X^2\right)\dot{\pi}^I \dot{\pi}^I \\
&+ \frac{1}{a^2}\left(-\frac{1}{3} F_X X - \frac{1}{3} F_{XX} X^2 - \frac{2}{9}\left(F_{XY} + F_{XZ}\right) X\right)(\partial_i \pi_j)^2 \\
&+ \frac{1}{a^2}\left(-\frac{2}{9} F_{XXX} X^3 - \frac{4}{9} F_{XX} X^2 - \frac{2}{27}\left(F_{XY} + F_{XZ}\right) X\right)(\partial_i \pi^i)^2\Bigg]\,.
\end{aligned} \qquad \text{(I.56)}$$

where the linear correction exactly cancels (I.52). Therefore, the full quadratic Lagrangian is

$$\begin{aligned}
\mathcal{L}_2 \sim (1-\alpha) a^2 \Bigg[&-\frac{1}{3} F_X X \dot{\pi}^I \dot{\pi}^I + \frac{1}{a^2}\left(\frac{1}{3} F_X X + \frac{2}{9}(F_Y + F_Z)\right)(\partial_i \pi_j)^2 \\
&+ \frac{1}{a^2}\left(\frac{2}{9} F_{XX} X^2 + \frac{2}{27}(F_Y + F_Z)\right)(\partial_i \pi^i)^2\Bigg]\,,
\end{aligned} \qquad \text{(I.57)}$$

where every term is evaluated on X_0. Therefore, the phonon speeds are still the same as in the original case.

I.6 Curvature perturbations

In our gauge, the two curvature perturbations are as follows:

$$\mathcal{R} = H \delta u\,, \qquad \text{(I.58)}$$

$$\zeta = -H \frac{\delta \rho}{\dot{\bar{\rho}}}\,, \qquad \text{(I.59)}$$

where $\delta T_{00} = -\bar{\rho}\delta g_{00} + \delta\rho$ and $\delta T_{i0} = \bar{p}\delta g_{i0} - (\bar{\rho}+\bar{p})(\partial_i \delta u + \delta u_i^V)$. With (I.28), the zeroth order terms in the energy-momentum tensor are [27]

$$\bar{T}_{00} = -\left[\left(1-\frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X\right]_b = -\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0}\,, \tag{I.60}$$

$$\begin{aligned}\bar{T}_{ij} &= a^2\left[\left(1-\frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X\right]_b \delta_{ij} \\ &\qquad - 2\left(1-\frac{\alpha}{4}\right)\delta_i^I\left(1-\frac{\alpha}{4}\right)\delta_j^J\left[(1-\alpha)F_X + \frac{\alpha}{2}F_X X\right]_b \delta^{IJ} \\ &= a^2\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - 2\left(1-\frac{3\alpha}{2}\right)F_X|_{X=X_0}\,.\end{aligned} \tag{I.61}$$

We thus have

$$\bar{\rho} = -\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0}\,, \tag{I.62}$$

$$\begin{aligned}\bar{p} &= \left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - \frac{2}{a^2}\left(1-\frac{3\alpha}{2}\right)F_X|_{X=X_0} \\ &= \left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - \left(1-\frac{3\alpha}{2}\right)\frac{2}{3}F_X X|_{X=X_0}\,.\end{aligned} \tag{I.63}$$

The first order terms in the energy-momentum tensor are

$$\begin{aligned}\delta T_{00} &= \delta\hat{g}'_{00}\left[\left(1-\frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X\right]_b - \left[(1-\alpha)\,F_X + \frac{\alpha}{2}F_X X\right]_b \delta X \\ &= \delta\hat{g}'_{00}\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - (1-\alpha)\,F_X|_{X=X_0}\delta X \\ \delta T_{i0} &= \delta\hat{g}'_{i0}(x')\left[\left(1-\frac{3\alpha}{2}\right)F + \frac{\alpha}{2}F_X X\right]_b - 2\partial'_0\pi'^I\partial'_i\phi'^I\left[(1-\alpha)\,F_X + \frac{\alpha}{2}F_X X\right]_b \\ &= \delta\hat{g}'_{0i}\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - 2\partial'_0\pi'^I\partial'_i\phi'^I\,(1-\alpha)\,F_X|_{X=X_0} \\ &= \delta\hat{g}'_{0i}\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - 2\,(1-2\alpha)\,\dot{\pi}^i F_X|_{X=X_0}\end{aligned} \tag{I.64}$$

[27]The subscript b means "evaluated on the background."

In regards to δT_{00}, the second term in δT_{00} is $\delta\rho$. We thus have

$$
\begin{aligned}
\delta\rho' &= -\left(1-\alpha\right) F_X|_{X=X_0}\delta X \\
&= -\left(1-\alpha\right) F_X|_{X=X_0}\frac{3}{a^2}\left(1-\frac{\alpha}{2}\right)\frac{2}{3}\left(1+\frac{\alpha}{4}\right)\partial_i'\pi'^i \\
&= -\left(1-\frac{3\alpha}{2}\right) F_X X|_{X=X_0}\frac{2}{3}\left(1+\frac{\alpha}{4}\right)\partial_i'\pi'^i \\
&= -\left(1-\frac{3\alpha}{2}\right) F_X X|_{X=X_0}\frac{2}{3}\partial_i\pi^i \qquad \because \partial_i' = \left(1-\frac{\alpha}{4}\right)\partial_i \text{ and } \pi'^I(x') = \pi^I(x) \\
&= -(\bar\rho+\bar p)\partial_i\pi^i\,.
\end{aligned} \tag{I.65}
$$

Let us now move on to δu.

$$
\begin{aligned}
\delta T_{i0} &= \delta\hat g_{0i}'\left(1-\frac{3\alpha}{2}\right)F|_{X=X_0} - 2\left(1-2\alpha\right)\dot\pi^i F_X|_{X=X_0} \\
&= \delta\hat g_{0i}'\bar p + \left(1-\frac{3\alpha}{2}\right)\left[\delta\hat g_{0i}'\frac{2}{3}F_X X|_{X=X_0} - 2\left(1-\frac{\alpha}{2}\right)\dot\pi^i F_X|_{X=X_0}\right]\,.
\end{aligned} \tag{I.66}
$$

We thus have

$$
\begin{aligned}
-(\bar\rho+\bar p)\partial_i'\delta u' &= \left(1-\frac{3\alpha}{2}\right)\frac{2}{3}F_X X|_{X=X_0}\partial_i'\delta u' \\
&= \left(1-\frac{3\alpha}{2}\right)\left[\delta\hat g_{0i}'\frac{2}{3}F_X X|_{X=X_0} - 2\left(1-\frac{\alpha}{2}\right)\dot\pi^i F_X|_{X=X_0}\right]\,, \qquad \text{(I.67)} \\
\partial_i'\delta u' &= \delta\hat g_{0i}' - a^2\left(1-\frac{\alpha}{2}\right)\dot\pi^i \\
&= \left(1+\frac{\alpha}{2}\right)a\partial_i B - \alpha a^2\dot\pi^i - a^2\left(1-\frac{\alpha}{2}\right)\dot\pi^i \\
&= \left(1+\frac{\alpha}{2}\right)\left[a\partial_i B - a^2\dot\pi^i\right] \\
&= \left(1-\frac{\alpha}{4}\right)\partial_i\delta u'\,, \qquad \text{(I.68)} \\
\delta u' &= \left(1+\frac{3\alpha}{4}\right)\left[aB - \frac{a^2}{\sqrt{-\nabla^2}}\dot\pi_L\right] \\
&= \left(1+\frac{3\alpha}{4}\right)\delta u\,. \qquad \text{(I.69)}
\end{aligned}
$$

Using these $\delta\rho'$ and $\delta u'$, we conclude that

$$\mathcal{R}' = H'\delta u' = \left(1 - \frac{3\alpha}{4}\right) H \left(1 + \frac{3\alpha}{4}\right) \delta u$$
$$= \mathcal{R}\,, \tag{I.70}$$
$$\zeta' = -H'\frac{\delta\rho'}{\dot{\bar{\rho}}} = -\frac{1}{6M_{\rm p}^2}\frac{\delta\rho'}{\dot{H}'} = \frac{1}{3}\frac{\delta\rho'}{\bar{\rho}+\bar{p}} = \frac{1}{3}\partial_i\pi^i$$
$$= \zeta\,, \tag{I.71}$$

where we used $\bar{\rho} = 3M_{\rm p}^2 H'^2$ and $\dot{\bar{\rho}} = 6M_{\rm p}^2 H'\dot{H}' = -3H'(\bar{\rho}+\bar{p})$. Both are invariant under the series of transformations.

I.7 Corrections to the power spectrum

Since γ and ζ do not change under the series of transformations,

$$\hat{\gamma}'_{ij}(x') = \gamma_{ij}(x)\,, \tag{I.72}$$
$$\zeta'(x') = \zeta(x)\,, \tag{I.73}$$

we can just replace proper physical variables with the new and 'hat' versions. Please keep in mind that these replacements should not include momentum k for the following reason. As an illustration, the definition of $\zeta(\vec{k})$ is

$$\zeta(\vec{k}) = \int d^3x\,\zeta(x)e^{-i\vec{k}\cdot\vec{x}} = \int d^3x'/\lambda^3\,\zeta(x'/\lambda)e^{-i\vec{k}\cdot\vec{x}'/\lambda}$$
$$\equiv \frac{1}{\lambda^3}\int d^3x'\,\zeta'(x')e^{-i\vec{k}\cdot\vec{x}'/\lambda} = \frac{1}{\lambda^3}\zeta'(\vec{k}/\lambda)\,. \tag{I.74}$$

where $\lambda \equiv (1+\alpha/4)$ and $\zeta'(x') \equiv \zeta(x'/\lambda)$. Therefore, the two-point function of $\zeta(\vec{k})$ is

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle = \frac{1}{\lambda^6}\langle\zeta'(\vec{k}_1/\lambda)\zeta'(\vec{k}_2/\lambda)\rangle = \langle\zeta'(\vec{k}_1)\zeta'(\vec{k}_2)\rangle\,. \tag{I.75}$$

Thus, we just need to replace other physical quantities such as H, c_L^2, etc, but not momentum $\vec{k}$. In [52], one has

$$\langle\gamma\gamma\rangle \propto \frac{H^2}{M_{\rm p}^2}\,, \tag{I.76}$$

$$\langle\zeta\zeta\rangle \propto \frac{1}{\epsilon c_L^5}\frac{H^2}{M_{\rm p}^2}\,. \tag{I.77}$$

Among the quantities in these spectra, only H^2 is changed by a factor of $1 - 3\alpha/2$. Therefore, we have

$$\langle\gamma\gamma\rangle = \left(1 - \frac{3\alpha}{2}\right)\langle\gamma\gamma\rangle_{\rm minimal}\,, \tag{I.78}$$

$$\langle\zeta\zeta\rangle = \left(1 - \frac{3\alpha}{2}\right)\langle\zeta\zeta\rangle_{\rm minimal}\,. \tag{I.79}$$

These results are indeed consistent with the exact computations done in Sect. 4.4.1 and 4.4.2, in which both power spectra are rescaled by $\frac{1}{(1+\alpha)^{3/2}}$, or $(1 - \frac{3\alpha}{2})$ in the small α limit.